Abstract

We study group competition with a single public good prize, perfectly discriminating contest success function, and the weakest-link effort technology, in which the marginal cost of effort for each player is private information. We focus on pure strategy Bayes-Nash equilibria and show that teammates always employ symmetric strategies. Various degrees of coordination are possible, ranging from all cost types coordinating on a single effort level to every cost type choosing a distinct effort level. Such coordination may not enhance welfare. If groups are symmetric except for group size, players in the smaller group bid more aggressively than those in the larger group, but when the asymmetries are along multiple dimensions, no clear-cut conclusions can be made with respect to the effects of group sizes and valuations. As an additional avenue for cooperation, we investigate incentives to share private information via cheap talk among teammates, who then coordinate on the effort level most preferred by the player with the largest cost. A single group sharing information does better. But when players within each group cooperate in this fashion, all within-group gains are lost to increased competition between groups.

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Declaration of interest: none
1 Introduction

Contests between groups are a ubiquitous economic phenomenon. Examples that span a wide range of environments include lobbying, R & D races, political competition, warfare, and sports, among others.

There are many dimensions to modeling a contest (Konrad, 2009). The winning group may be awarded a single prize or multiple prizes, which could be a public good consumed nonexclusively by all members, a private good, or a mixture of the two. Depending on how monolithic the competing entities are, one can distinguish between individualistic and group contests. In modeling group contests, a central issue is the interaction between efforts of its members in determining the group performance. To capture different degrees of complementarity of efforts within a group, several types of the effort aggregation function have been introduced, such as the “best shot” (a group’s effort equals the highest effort within the group), the “weakest link” (a group’s effort equals the lowest effort within the group), and the additive aggregation function (a group’s effort equals the sum of efforts within the group, so that efforts of the group members are perfect substitutes). Furthermore, based on the rule that determines the winner (called the contest success function, or csf), one can model a contest as perfectly or imperfectly discriminating. The former is based upon the auction-type csf, while the latter is epitomized by a Tullock contest.

We model group competition as a weakest-link perfectly discriminating contest with a public good prize. Such strong complementarity arises naturally in sports contests such as synchronized swimming or diving. Competitions between chamber orchestras or barbershop quartets also provide examples in which an entire performance can be ruined by the lackluster effort of a single out-of-tune member. Similarly, an interdisciplinary research proposal may require effort among several dimensions, each addressed by a subject-area specialist in a group, and a serious flaw in even a single dimension may suffice to doom the entire proposal, thus creating strong complementarity among individual efforts. Other examples of the weakest link include supply chains and O-ring products.

At the focus of our study is the interplay between the weakest-link effort technology and players being privately informed about their marginal costs of effort while having only a vague idea of their teammates’ and rivals’ costs. With the weakest-link technology some players’ efforts within a group may be wasted due to a lower contribution by a teammate. This suggests potential gains from equilibrium coordination. One expects a single coordinating group to benefit, so we focus on symmetric coordination across groups. We concentrate on pure strategy Bayes-Nash equilibria, show that teammates always employ symmetric strategies, and demonstrate that a wide range of coordination is possible, from all cost types coordinating on a single effort level to every cost type choosing a distinct effort level. However, we show that increasing the degree of coordination may improve or worsen welfare, and we provide a sufficient condition for each
outcome to occur.

Similarly, one may expect coordination to be more difficult in larger groups. To analyze this issue, we explore the effects of symmetrically increasing group sizes. Again, this may increase or decrease players' welfare and we are able to provide a sufficient condition for each outcome to occur. Interestingly, this condition is identical to the one we found relating the degree of equilibrium coordination within each group with welfare in the contest.

We extend the analysis to asymmetric contests and find that, if the only difference among groups is size, then players in the smaller group are more aggressive. But if asymmetries arise along multiple dimensions (e.g., group sizes and cost distributions), then no clear-cut conclusions can be made.

Finally, as an additional way to explore cooperation within groups, we look at teammates' incentives to exchange private information via cheap talk. We find that, although welfare is always higher in the cooperating group provided that its competitors are not engaged in (within-group) information sharing, all gains are lost to increased competition between groups when cooperation is symmetric.

Although group contests with complete information\(^1\), as well as individualistic contests with private information\(^2\) have been studied extensively, issues surrounding player-level private information\(^3\) and especially information exchange in group contests remain under-researched. While Brookins and Ryvkin (2015) and Einy et al. (2015) derive some general existence results for a broad class of group contests with private information, we focus on the properties of equilibria and look into whether private information hampers or facilitates players’ ability to coordinate in the presence of the weakest-link technology. Papers most closely related to our research are Barbieri and Malueg (2016) and Chowdhury et al. (2016). The former analyzes best-shot competition, the latter full information environments, so the differences with our setup are clear.

To the best of our knowledge, our study is the first to look at players’ incentives to exchange private information via cheap talk in a group contest setting.\(^4\) Information sharing in individualistic contests is examined inter alia by Kovenock et al. (2015), who find that exchange of private information among competing entities unambiguously decreases welfare.

The rest of the paper is organized as follows. In the next section, we describe our model, describe the conditions that equilibrium effort distributions must satisfy, and characterize equilibria with a finite effort support. Equilibria with infinite supports are characterized in Section 3 for symmetric groups and in Section 4 for the general setup in which groups can be asymmetric. We explore the possibility of information

\(^1\) See e.g., Baik et al. (2001), Baik (2008), Chowdhury et al. (2013), Barbieri et al. (2014), Topolyan (2014), Chowdhury and Topolyan (2016), and Chowdhury et al. (2016).

\(^2\) See e.g., Morath and Münster (2008), Parreiras and Rubinchik (2010), Ryvkin (2010), Kirkegaard (2013a) and (2013b), and Wasser (2013a) and (2013b).

\(^3\) Eliaz and Wu (2016) study imperfectly discriminating group contests with group-level private information, where the prize value, albeit stochastic, is the same for all group members.

\(^4\) Barbieri and Malueg (2018) study information exchange via cheap talk in a model of public good provision.
sharing within a group via cheap talk in Section 5. Section 6 concludes.

2 The model

Suppose two groups compete for a single prize in an all-pay auction. The prize is a public good that is consumed nonrivalrously and nonexclusively by all members of the winning group. Group \( l \) has \( n_l \) members who share a common valuation of the prize, \( v_l, l = 1, 2 \). Denote by \( \mathcal{I}_l \) the index set of players in group \( l \). Effort cost is a private information characteristic, independently distributed across players. Within each group \( l \) we assume the marginal cost of effort for each player is drawn according to the same atomless cumulative distribution (cdf) \( F_l \) over the interval \([c, \bar{c}]\), where \( c \geq 0 \); furthermore, we assume that \( F_l \) possesses density \( f_l \), which is strictly positive on \((c, \bar{c})\), \( l = 1, 2 \). Distributions, valuations, and group sizes are common knowledge. Each player is informed of the realization of her own cost, but has no information about the realizations of others’ cost, even of fellow group members.

Within each group, players’ individual efforts are transformed into the group effort via the weakest-link effort technology; that is, the group effort equals the minimum effort exerted by members in the group. The group with the larger weakest-link effort wins the contest; ties are broken with equal probability in favor of each group. Regardless of victory or defeat, each player sustains the full cost of her effort. Therefore, the realized payoff to a member of group \( l \) with cost \( c \) exerting effort \( x \) is \( v_l - cx \) if her group wins the contest and \(-cx\) if it loses.

In what follows we focus on Bayes-Nash equilibria in pure strategies.\(^5\)

**Definition 1.** A pure strategy of player \( i \) in group \( l \) is an \( F_l \)-measurable function \( g_{li}(c_{li}) \) that prescribes to every realized cost of effort \( c_{li} \) the corresponding effort level \( g_{li}(c_{li}) \).

Note that a pure strategy \( g_{li} \) generates the bidding distribution \( H_{g_{li}} \) on \( \mathbb{R}_+ \) as follows:

\[
H_{g_{li}}(x) = \mu_{F_l} \left[ c_{li} : g_{li}(c_{li}) \leq x \right],
\]

where \( \mu_{F_l} \) is the probability measure on \([c, \bar{c}]\) generated by \( F_l \). Since \( g_{li} \) is \( F_l \)-measurable, \( H_{g_{li}} \) is well-defined.

In what follows, if the strategy \( g_{li} \) is part of an equilibrium and confusion does not arise, we simplify notation and denote \( H_{g_{li}} \) with \( H^i_l \). Similarly, we denote the equilibrium cdf of the overall lowest effort in group \( l \) with \( H_l \). Furthermore, we indicate with \( H_l^{-\{i\}} \) the equilibrium cdf of the lowest effort among all members of group \( l \) excluding player \( i \). Similarly, \( H_l^{-\{i,j\}} \) excludes players \( i \) and \( j \). Denote the top end of \( F_l \)

\(^5\)We use the standard notion of Bayes-Nash equilibrium (see e.g., MWG ch. 8). Standard arguments show that equilibrium strategies must be non-increasing in cost.
the support of $H_l$ with $x_l$ and the bottom end with $\bar{x}_l$, $l = 1, 2$; that is, $x_l = \min\{x \mid x \in \text{supp} H_l\}$ and $\bar{x}_l = \max\{x \mid x \in \text{supp} H_l\}$.

For concreteness, consider a member $i$ of group 1; calculations for group 2 follow similarly. With a contribution of $\gamma < \bar{x}$, player $i$ with cost $c$ obtains equilibrium interim utility

$$U_{1i}(\gamma; c) \equiv v(1 - H_1^{-1}(\gamma)) H_2(\gamma) + v \int_0^\gamma H_2(s) dH_1^{-1}(s) - c\gamma. \quad (1)$$

Given the behavior of the other players, player $i$ with cost $c$ would now choose $\gamma$ to maximize $U_{1i}(\gamma; c)$.

**Definition 2.** A Bayes-Nash equilibrium is called **degenerate** if the supports of $H_1$ and $H_2$ are singletons, i.e., all types in a group bid the same amount. It is called **semi-degenerate** if one group’s support is a singleton while the other group’s support is not. Otherwise, it is called **non-degenerate**.

We begin by characterizing degenerate and semi-degenerate equilibria. Thereafter, we study our topic of primary interest, the non-degenerate equilibria. The following result is an extension of Theorem 2.8 of Chowdhury et al. (2013); the proof is immediate and here omitted. The idea is simply that at degenerate equilibria all players coordinate on an effort that the highest-cost player in the contest would be willing to exert. While other types would be willing to exert greater effort, the weakest-link aggregation rule prevents such deviations from being worthwhile.

**Theorem 1** (Degenerate equilibria). Degenerate equilibria exist if and only if both $n_1$ and $n_2$ are at least 2.

Now suppose $n_1 \geq 2$ and $n_2 \geq 2$. All degenerate symmetric Bayes-Nash equilibria are as follows: $g(c) \equiv \lambda$, where $0 \leq \lambda \leq \min\left\{ \frac{v_1}{2\bar{c}}, \frac{v_2}{2\bar{c}} \right\}$.

The following theorem characterizes all semi-degenerate equilibria (this and most subsequent proofs are in the Appendix). Note that members of the team not using a degenerate strategy all use the same strategy.

**Theorem 2** (Semi-degenerate equilibria). Semi-degenerate equilibria exist if and only if both $n_1$ and $n_2$ are at least 2 and $v_1 \neq v_2$. Now suppose $n_1, n_2 \geq 2$, let $v_1 > v_2$ without loss of generality, and let $a^*$ be the unique solution to $a^*F_2^{-1}(a^*) = \frac{v_2 - \bar{c}}{v_1}$. The set of all semi-degenerate equilibria is the following. Fix $a \in [a^*, 1)$. Then,

- In group 1, each player contributes $\bar{x} = \frac{v_2 - \bar{c}}{F_2^{-1}(a)}$ regardless of her cost type.
- In group 2, cost types $c < F_2^{-1}(a)$ contribute $\bar{x}$ while cost types $c > F_2^{-1}(a)$ contribute 0; cost type $c = F_2^{-1}(a)$ is indifferent between contributing $\bar{x}$ and 0.

While degenerate and semi-degenerate equilibria are easy to describe, we now establish general properties of non-degenerate equilibrium. We begin by establishing properties of the cdfs $H_1$ and $H_2$. 
Lemma 1 (Necessary conditions for non-degenerate equilibrium cdfs). In any non-degenerate Bayes-Nash equilibrium, the pdfs of groups’ weakest-link efforts, $H_1$ and $H_2$, display these properties:

1. $x_1 = x_2 = 0$ and $\bar{x}_1 = \bar{x}_2 = \bar{x} > 0$, for some $\bar{x} > 0$.

2. There is no mass point for $H_1$ or $H_2$ in $(0, \bar{x})$. Moreover, not more than one group can have a mass point at zero.

3. $H_1$ and $H_2$ have the same support $S$.

4. $S$ contains no “holes” except possibly at the top, i.e., $[0, \bar{x}] \setminus S = (a, \bar{x})$ for some $a \leq \bar{x}$. Furthermore, one distribution admits a mass point at $\bar{x}$ if and only if the same is true for the other distribution and $a < \bar{x}$.

Lemma 1 implies $H_1$ and $H_2$ are continuous and strictly increasing on $[0, \bar{x})$, which in turn implies that individual strategies are strictly decreasing when taking values in $(0, \bar{x})$. It turns out this is sufficient to establish that teammates use essentially the same strategy.\footnote{The conclusion of Lemma 2 also applies to degenerate and semi-degenerate equilibria.}

Lemma 2 (Equilibrium strategies are symmetric within a team). Fix a group $l$, $l = 1, 2$. In any non-degenerate Bayes-Nash equilibrium, for any two agents $i$ and $j$ in group $l$, we have $g_{li} = g_{lj} = g_l$, except on a set of measure zero.

3 Symmetric teams

We now focus on the case in which $n_1 = n_2 = n$, $v_1 = v_2 = v$, and $F_1 = F_2 = F$ and characterize all nondegenerate equilibria. By Lemmas 1 and 2 this requires identifying two strategies, $g_1$ and $g_2$, respectively, for members of groups 1 and 2. We begin by showing, in fact, that players use common strategies, that is, $g_1 = g_2$.

Lemma 3 (Nondegenerate equilibria are symmetric). In any non-degenerate Bayes-Nash equilibrium, we have $g_{li} = g_l$ for some common function $g$, except possibly on a set of measure zero, $i = 1, \ldots, n$ and $l = 1, 2$.

The following corollary of Lemmas 1 and 2 establishes additional properties of $g$ in a non-degenerate equilibrium; the proof is immediate and has been omitted.

Corollary 1. If $g$ is a symmetric non-degenerate Bayes-Nash equilibrium strategy, we have the following:

1. $g(\bar{c}) = 0$. 

6The conclusion of Lemma 2 also applies to degenerate and semi-degenerate equilibria.
2. $g$ admits no mass at $z \in [0, \bar{x})$, where $\bar{x} = g(\xi)$.

3. $g$ is continuous on $(c_0, \xi]$, where $c_0 = \sup \{c : g(c) = \bar{x}\}$.

Corollary 1 implies that a symmetric equilibrium bidding strategy is continuous, with the possible exception of a jump to a flat spot in the vicinity of $\xi$, which would place a mass at $\bar{x}$. We consider in turn these equilibria without and with a mass point.

### 3.1 The symmetric equilibrium strategy without mass

**Theorem 3.** There exists a unique non-degenerate Bayes-Nash equilibrium without mass at the upper bound. It is symmetric and has strategy given by

$$g(c) = vn \int_c^{\bar{c}} \frac{F(\tau)^{2(n-1)} f(\tau)}{\tau} d\tau \text{ for all } c \leq \bar{c}. \tag{2}$$

**Proof.** First, note that Lemma 3 implies that any non-degenerate equilibrium is symmetric. The existence part of the proof is by construction. Let $g$ be the symmetric equilibrium strategy. Let $c_1^M = \max\{c_{12}, ..., c_{1n}\}$ denote a random variable which is the maximum of the random variables $c_{12}, ..., c_{1n}$. Note that the relevant distributions for $c_1^M$ are the cdf $Q(c_1^M) = [F(c_1^M)]^{n-1}$ and the density $q(c_1^M) = (n - 1) [F(c_1^M)]^{n-2} f(c_1^M)$.

The payoff of player 1 with marginal cost $c_{11}$ that acts like type $c_1^{\bar{a}}$ can then be expressed as

$$V_{11}(c_1^{\bar{a}}, c_{11}) = \int_{\xi}^{c_1^{\bar{a}}} v \left(1 - (F(c_1^{\bar{a}}))^n\right) q(c_1^M) dc_1^M + \int_{c_1^{\bar{a}}}^{\bar{c}} v \left(1 - (F(c_1^M))^n\right) q(c_1^M) dc_1^M - c_{11} g(c_1^{\bar{a}})$$

$$= \frac{n}{2n-1} \left(1 - (F(c_1^{\bar{a}}))^{2n-1}\right) v - c_{11} g(c_1^{\bar{a}}). \tag{3}$$

The first expression in (3) is easily interpreted as the interim probability that group 1 wins when player 1 in group 1 acts as if her cost is $c_1^{\bar{a}}$. If player 1 in group 1 exerts least effort, her group will not win. The probability that one of the $2n - 1$ players other than player 1 in group 1 has marginal cost exceeding $c_1^{\bar{a}}$ is $1 - (F(c_1^{\bar{a}}))^{2n-1}$. Given this, group 1 wins if the highest-cost player is in group 2, which happens with probability $\frac{n}{2n-1}$, since players act symmetrically. Now, taking the derivative of (3) with respect to the type $c_1^{\bar{a}}$, we have

$$\frac{\partial V_{11}(c_1^{\bar{a}}, c_{11})}{\partial c_1^{\bar{a}}} = -nv (F(c_1^{\bar{a}}))^{2(n-1)} f(c_1^{\bar{a}}) - c_{11} g'(c_1^{\bar{a}}).$$

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7 We say $g$ admits mass at $z$ whenever there exist $c_1, c_2$ such that $c_1 < c_2$ and $g(c) = z$ for all $z \in (c_1, c_2)$.

8 It should be clear that when $g$ admits no mass at $\bar{x}$, it is continuous everywhere, in which case the support of $H$ does not have “holes.” When $g$ does admit mass at $\bar{x}$, there is a hole in the support of $H$ equal to $(a, \bar{x})$ for some $a < \bar{x}$. 

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Thus, the FOC, evaluated at $c_{a11} = c_{11}$, yields

$$g'(c_{11}) = -\frac{nv}{c_{11}} [F(c_{11})]^{2(n-1)} f(c_{11}).$$

Now $g(\cdot)$ is derived by integrating this equation from $c$ to $\bar{c}$ and using the boundary condition $g(\bar{c}) = 0$.

To see that the FOC is sufficient, note that deviations to contributions larger than $g(c)$ are never profitable because they increase cost and do not increase the probability of winning. Moreover, using the FOC, we have

$$\frac{\partial V_{11}(c_{a11}, c_{11})}{\partial c_{a11}} = -nF(c_{a11})f(c_{11}) - c_{11}g'(c_{11}) = nv[F(c_{11})]^{2(n-1)} f(c_{11}) \left( \frac{c_{11}^{2(n-1)} - 1}{c_{a11}^{2(n-1)}} \right),$$

which is positive for $c_{a11} < c_{11}$ and negative for $c_{a11} > c_{11}$, implying that the solution of the FOC identifies a best response. Necessity of the FOC for the maximum implies that the constructed equilibrium is unique within the class of symmetric non-degenerate equilibria without mass at the upper bound, which, combined with Lemma 3, implies uniqueness within the class of non-degenerate equilibria without mass at the upper bound.

**Example 1.** Suppose $F(c) = c^a$ on $[0, 1]$, where $a > 0$. Then we have

$$g'(c) = -\frac{nv}{c} \times c^{2(n-1)a} \times ac^{a-1} = -nac^{(2n-1)a-2},$$

yielding, for $a \neq 1/(2n - 1)$,

$$g(c) = K - \frac{nav}{(2n-1)a-1} \times c^{(2n-1)a-1}.$$

The constant of integration $K$ must satisfy $g(1) = 0$, so $K = \frac{nav}{(2n-1)a-1}$. Thus,

$$g(c) = \frac{nav}{(2n-1)a-1} \left( 1 - c^{(2n-1)a-1} \right).$$

If $a = 1/(2n - 1)$, then $g(c) = -n \log(c)/(2n - 1)$. Depending on the parameters $a$ and $n$, the equilibrium strategy may be concave, linear, or convex in $c$.

It is natural to ask, “How does expected utility change as the number of agents per group increases?” In any symmetric equilibrium, each group’s ex ante payoff gross of costs is $v/2$. Therefore, an equivalent question is: “how does expected cost change with $n$?” On the one hand, within-group free-riding becomes more
intense in a larger group. On the other hand, if free-riding plagues both groups symmetrically, perhaps the competitiveness of the contest is reduced and individual contestants benefit. It turns out that in Example 1 these two effects exactly cancel out. Indeed, we have

$$\int_{c_l}^{c_0} cg(c)f(c) dc = \frac{nav}{(2n-1)a-1} \int_0^1 \left(1 - c^{(2n-1)a-1}\right) ac^n dc = \frac{av}{2(a+1)},$$

which is independent of the number of agents per group $n$. The following proposition shows sufficient conditions such that either effect dominates.

**Proposition 1 (Effects of increasing group size).** Consider the symmetric equilibrium described in Theorem 3. Increasing the number of team members $n$ per team decreases (increases) each agent’s expected utility if

$$W(\tau) = \frac{\int_{c_0}^{c_0} cf(c)dc}{\tau F(\tau)}$$

is increasing (decreasing) <constant> in $\tau$.

### 3.2 The symmetric equilibrium strategy with mass at the maximum effort

We now explore the possibility of equilibrium effort distributions with a mass point at the top. It turns out that, varying the size of this mass point, one links the degenerate equilibria in Theorem 1, in which all agents coordinate on the same effort, with the equilibrium in Theorem 3, in which agents with a different marginal cost realization choose a different effort level. Therefore, the size of the mass point can be thought as a convenient parameter that describes the degree of coordination within teams.

Let $g$ be the symmetric equilibrium strategy such that all types in $[c, c_0)$ for some $c_0 > c$ exert the maximum effort $\bar{x}$ (to be determined), $g$ exhibits a jump discontinuity at $c_0 \in (c_l, \bar{c})$ and is continuous on $(c_0, \bar{c}]$. Define $x' = \lim_{c \to c_0^+} g(c)$, so that $x' < \bar{x}$. As before, $c_1^M = \max\{c_{12}, ..., c_{1n}\}$, so that the relevant distributions for $c_1^M$ are the cdf $Q(c_1^M) = [F(c_1^M)]^{n-1}$ and the density $q(c_1^M) = (n-1) [F(c_1^M)]^{n-2} f(c_1^M)$.

On $(c_0, \bar{c}]$ the equilibrium strategy coincides with that derived in the previous subsection because the (greater) efforts of lower-cost players are not determinative. If all types in $[c, c_0)$ exert effort $\bar{x}$, then for $n \geq 2$ these types have no incentive to increase effort above $\bar{x}$. Moreover, they are willing to exert exactly effort $\bar{x}$ if the type $c_0$ player is just indifferent between $\bar{x}$ and $x'$. The relevant interim payoffs for the type-$c_0$ player are

$$U_1(x'; c_0) = \frac{nv}{2n-1} \left(1 - (F(c_0))^{2n-1}\right) - c_0 x'$$

(by (3))
and

\[ U_1(\bar{x}; c_0) = v [F(c_0)]^{n-1} \left( 1 - \frac{1}{2} [F(c_0)]^n \right) + \int_{c_0}^{\bar{x}} v \left( 1 - (F(c_1^M))^n \right) q(c_1^M) dc_1^M - c_0 \bar{x}, \]

where the latter equation uses the tie-breaking rule that if all players choose effort \( \bar{x} \), then each group has a 1/2 chance of winning. Indifference for a type-\( c_0 \) player implies

\[ 0 = U_1(x'; c_0) - U_1(\bar{x}; c_0) = c_0 (\bar{x} - x') - \frac{v}{2} (F(c_0))^{2n-1}. \]

Note that in equilibrium \( g \) could be either left- or right-continuous, with \( g(c_0) \) equalling either \( x' \) or \( \bar{x} \). For any \( c \in (c_0, \bar{c}) \) the equilibrium strategy is given by (2), and

\[ x' = \lim_{c \downarrow c_0} g(c) = vn \int_{c_0}^{\bar{x}} \frac{F(\tau)^{2(n-1)}f(\tau)}{\tau} d\tau. \]

Finally, \( \bar{x} \) is determined from (5):

\[ \bar{x} = vn \int_{c_0}^{\bar{x}} \frac{F(\tau)^{2(n-1)}f(\tau)}{\tau} d\tau + \frac{v}{2c_0} [F(c_0)]^{2n-1}. \]

We thus have the following result.

**Theorem 4.** Fix \( n \geq 2 \). There is a continuum of non-degenerate Bayes-Nash equilibria, indexed by \( c_0 \in (\underline{c}, \bar{c}) \), with mass at the upper bound. Given any \( c_0 \in (\underline{c}, \bar{c}) \), any cost type \( c_0 < c \leq \bar{c} \) contributes

\[ g(c) = vn \int_{c_0}^{c} \frac{F(\tau)^{2(n-1)}f(\tau)}{\tau} d\tau. \]

Types in \( [\underline{c}, c_0) \) contribute

\[ \bar{x} = vn \int_{c_0}^{\bar{x}} \frac{F(\tau)^{2(n-1)}f(\tau)}{\tau} d\tau + \frac{v}{2c_0} [F(c_0)]^{2n-1}, \]

while type \( c_0 \) is indifferent between contributing \( \bar{x} \) and \( x' \).

Thus, equilibrium mass on \( \bar{x}(c_0) \) increases from 0 to 1 as \( c_0 \) increases from \( \underline{c} \) to \( \bar{c} \). Notice that as \( c_0 \) approaches \( \underline{c} \), the equilibrium converges to the one of Theorem 3, while as \( c_0 \) approaches \( \bar{c} \), the equilibrium converges to the degenerate equilibrium where all cost types contribute \( \frac{v}{2\bar{c}} \). Figure 1 illustrates the nature of equilibria described by Theorems 3 and 4—linearity is inessential.
We first ask whether players benefit from playing a “jump equilibrium” rather than the strictly decreasing one. For $c > c_0$ there is no change in interim utilities since (2) and (7) are the same and the equilibrium behavior of lower-cost teammates is irrelevant. However, for $c < c_0$, interim utility in the jump equilibrium is

$$V(c; c_0) = \int_c^{c_0} \bar{x}(c_0) \, dc + \int_{c_0}^{\bar{c}} g(c) \, dc,$$

using the Envelope Theorem and the fact that the interim utility of type $\bar{c}$ is zero. Therefore, from Figure 1 it is clear that all types $c \in [\hat{c}, c_0)$ prefer the jump equilibrium to the strictly decreasing one. However, types near $\hat{c}$ may or may not prefer the jump equilibrium. But if type $c$ prefers the jump equilibrium, then so do all types $c < c_0$.

More generally, one may ask, “How does ex ante expected utility change as $c_0$ changes?” In a symmetric equilibrium, this is equivalent to asking: “how does expected cost change with $c_0$?” Defining the expected effort cost as $EC(c_0)$ and the largest effort as $\bar{x}(c_0)$ we have

$$EC(c_0) = \int_{\hat{c}}^{c_0} c\bar{x}(c_0) f(c) \, dc + \int_{c_0}^{\bar{c}} cg(c) f(c) \, dc;$$

in the Appendix, we verify that

$$EC'(c_0) = \frac{\nu}{2} [F(c_0)]^{2n-2} \left[ f(c_0) F(c_0) - \left( \frac{f(c_0)}{c_0} + \frac{F(c_0)}{(c_0)^2} \right) \int_{\hat{c}}^{c_0} cf(c) \, dc \right].$$

It turns out that for the power distributions in Example 1 we have $EC'(c_0) = 0$. Indeed, if $F(c) = c^a$, then $\int_{\hat{c}}^{c_0} cf(c) \, dc = \frac{a}{a+1} c_0^{a+1}$, $\frac{f(c_0)}{c_0} + \frac{F(c_0)}{(c_0)^2} = (a+1)c_0^{a-2}$, and $f(c_0) F(c_0) = ac_0^{2a-1}$, so in (9) the term in square brackets equals zero.

**Proposition 2** (Effects of increasing mass at the upper bound). Consider the class of equilibria described
in Theorem 4. Increasing the size of the mass point by increasing \( c_0 \) decreases (increases) <leaves constant> each agent’s expected utility if \( W(\tau) \), defined in (4), is increasing (decreasing) <constant> in \( \tau \).

The reason for the different outcomes in Proposition 2 is that increasing \( c_0 \) brings about two countervailing changes in equilibrium strategies. Because \( \bar{x}(c_0) \) is decreasing in \( c_0 \), if the jump point is increased from \( c_0 \) to \( c_0' \), then the contribution of types immediately to the left of \( c_0' \) is increased, while that of types closer to \( c \) is decreased. One may trace the effects of these two changes in interim utilities. From (8) we have

\[
\frac{\partial V(c; c_0)}{\partial c_0} = \bar{x}(c_0) - g(c_0) + \bar{x}'(c_0)(c_0 - c)
\]

\[
= \frac{v}{2c_0} [F(c_0)]^{2n-1} - (c_0 - c) \left[ \frac{v}{2c_0} [F(c_0)]^{2n-2} f(c_0) + \frac{v}{2(c_0)^2} [F(c_0)]^{2n-1} \right]
\]

\[
= \frac{v}{2c_0} [F(c_0)]^{2n-1} \left[ \frac{c}{c_0} - \frac{c_0 - c}{F(c_0)} f(c_0) \right].
\]

as expected, types close to \( c_0 \) benefit from the increase in \( c_0 \), but types close to \( c \) may be damaged.

4 Asymmetric teams: different values, sizes, or distributions

We now consider the general case in which \( n_1 \) and \( n_2 \), \( v_1 \) and \( v_2 \), and \( F_1 \) and \( F_2 \) are not necessarily the same and characterize all nondegenerate equilibria. By Lemmas 1 and 2 this requires identifying two strategies, \( g_1 \) and \( g_2 \); in equilibrium, all members of group \( i \) use \( g_i \), \( i = 1, 2 \). In contrast with the symmetric case, now one group can have a mass point at 0 in its distribution of minimum effort. And, as in the symmetric case, mass points at the largest effort remain possible. In our derivations, we use a construction due to Amann and Leininger (1996). For the range of efforts where both \( g_1 \) and \( g_2 \) are strictly decreasing, consider the function \( \xi(c_{11}) = g_2^{-1}(g_1(c_{11})) \), which maps any cost type \( c_{11} \) of player 1 in group 1 who puts effort \( g_1(c_{11}) \) into that cost type of player 1 in group 2 who exerts the same effort, i.e., \( c_{21} = \xi(c_{11}) = g_2^{-1}(g_1(c_{11})) \). Note that since both \( g_1 \) and \( g_2 \) are strictly decreasing, \( \xi \) is strictly increasing and, by the chain rule,

\[
\xi'(c) = (g_2^{-1})'(g_1(c))g_1'(c).
\]

Mass points at zero or at the largest contribution affect the range of cost types where \( \xi \) is well-defined. We consider first equilibrium strategies that do not put mass at the top, then those that do.

4.1 Equilibrium strategies without mass at the maximum effort

The following result, with proof in the Appendix, characterizes equilibrium.

\footnote{This is easily deduced from (6).}
Theorem 5. Assume \( \lim_{x \to \xi} f_2(x) \neq 0 \). There exists a unique non-degenerate Bayes-Nash equilibrium without mass at the upper bound.

Let \( \xi \) be the unique strictly increasing solution to the differential equation

\[
\xi'(c) = \frac{n_1 v_2 f_1(c)}{n_2 v_1 \xi(c) f_2(\xi(c))},
\]

(11) 

Together with the initial condition \( \xi(c) = \xi \). Each player in group 1 that exerts a positive effort uses strategy

\[
g_1(c) = -\int_c^\xi g_1'(\tau) d\tau,
\]

(12) 

where \( g_1'(\tau) = 0 \) if \( g_1(\tau) = 0 \), and otherwise

\[
g_1'(\tau) = -\frac{n_1 v_2 F_2(\xi(\tau))^{n_2 - 1} F_1(\tau)^{n_1 - 1}}{\xi(\tau)} f_2(\xi(\tau)).
\]

(13) 

Each player in group 2 that exerts positive effort uses strategy

\[
g_2(c) = g_1(\xi^{-1}(c)).
\]

(14) 

Theorem 5 does not describe which team, if any, puts positive probability on zero or how large the atom of probability is. These considerations are resolved by the solution to the differential equation (11) with initial condition \( \xi(c) = \xi \). For example, if in tracing this solution one finds a \( c' < \xi \) such that \( \xi(c') = \xi \), this means that group-1-player types in \( [c', \xi] \) never win; therefore, these types exert zero effort. And if instead \( \xi(\xi) = \xi'' \) for some \( \xi'' < \xi \), then it is group-2-player types \( [\xi'', \xi] \) that exert zero effort. The following proposition presents a sufficient condition for each case, in addition to determining which group contains the more “aggressive” players.

Proposition 3. The equilibrium described in Theorem 5 has the following properties:

1. If \( \forall c \in [\xi, \xi] \) we have \( \frac{n_1}{n_2} < \frac{v_1}{v_2} \cdot \frac{f_1(c)}{f_2(c)} \), then \( g_1(c) < g_2(c) \) \( \forall c \in (\xi, \xi) \). Every player in group 1 puts mass at zero, i.e., \( H_1^i(0) > 0 \) \( \forall i \in I_1 \), while \( H_2^j \) is atomless for any \( j \in I_2 \).

2. If \( \forall c \in [\xi, \xi] \) we have \( \frac{n_1}{n_2} = \frac{v_1}{v_2} \cdot \frac{f_1(c)}{f_2(c)} \), then \( g_1(c) = g_2(c) \) and neither group puts mass at zero.

3. If \( \forall c \in [\xi, \xi] \) we have \( \frac{n_1}{n_2} > \frac{v_1}{v_2} \cdot \frac{f_1(c)}{f_2(c)} \), then \( g_2(c) < g_1(c) \) \( \forall c \in (\xi, \xi) \). Further, \( H_1^i \) is atomless for any \( i \in I_1 \), while players in group 2 put mass at zero.

Notice that, assuming identical distributions \( F_1 \) and \( F_2 \), if \( n_1 v_2 = n_2 v_1 \), then \( \xi \) becomes the identity map, and every player uses the same strategy in equilibrium as in Theorem 3. Moreover, if distributions and
valuations are identical, then Proposition 3 shows that players in the smaller group bid more aggressively than those in the larger group. Surprisingly, this result is analogous to the one in Barbieri et al. (2013), who find that, in the semi-symmetric equilibria of the group best-shot all-pay auction with complete information and symmetric valuations, players in the larger group put mass at zero.

We now illustrate the results in this section with two examples. The next one applies Theorem 5 and Proposition 3, and further determines which group is more likely to win.

**Example 2.** $F_1$ and $F_2$ are uniform on $[0, 1]$, $n_1 = 1$, and $n_2 = 2$. We use Theorem 5 to find the equilibrium without mass at the maximum effort. Here, (11) reads as

$$\xi(c)\xi'(c) = \frac{v_2}{2v_1} c,$$

which can be solved as

$$\xi(c) = \sqrt{\frac{v_2}{2v_1}} c.$$

Consistent with Proposition 3, we see that equilibrium depends on the relation between $v_2$ and $2v_1$.

**CASE 1:** $v_2 \leq 2v_1$. If $v_2 < 2v_1$, then $\xi(1) < 1$, so group-2 players drop out for costs sufficiently close to 1, while the group-1 effort distribution is atomless. Therefore, $g_1'(c) < 0 \forall c \in [0, 1)$, and using (12), (13), and (14), we obtain $g_1(c) = v_2(1 - c)$ and

$$g_2(c) = \begin{cases} \sqrt{2v_1v_2} \left(\sqrt{\frac{v_2}{2v_1}} - c\right) & \text{if } c \leq \sqrt{\frac{v_2}{2v_1}} \\ 0 & \text{if } c > \sqrt{\frac{v_2}{2v_1}}. \end{cases}$$

Note that if $v_2 = 2v_1$, then members of groups 1 and 2 end up using the same strategy. If $v_2 < 2v_1$, then $g_2(c) < g_1(c)$ for all $c \in (0, 1)$, in accordance with Proposition 3. In either case, since $g_2(c) \leq g_1(c)$ and $n_2 > n_1$, it follows that group 1 is more likely to win.

**CASE 2:** $v_2 > 2v_1$. Proposition 3 indicates now that it is player 1 who drops out before her cost reaches 1. In particular, since $\xi(c) = \sqrt{\frac{v_2}{2v_1}} c$, types above $\sqrt{\frac{2v_1}{v_2}}$ do not exert effort. For these types, $g_1'(c) = 0$, so now (12), (13), and (14) yield

$$g_1(c) = \begin{cases} v_2 \left(\sqrt{\frac{2v_1}{v_2}} - c\right) & \text{if } c \leq \sqrt{\frac{2v_1}{v_2}} \\ 0 & \text{if } c > \sqrt{\frac{2v_1}{v_2}} \end{cases}$$

and $g_2(c) = \sqrt{2v_1v_2}(1 - c)$. Note that here we have $g_2(c) > g_1(c)$ for all $c \in (0, 1)$, in accordance with
Proposition 3. Nevertheless, it will still be the case that group 1 wins more often than group 2 when \( v_2 \) is not too much larger than \( 2v_1 \).

We conclude this section with a second example, which illustrates how to proceed when Proposition 3 does not apply. In particular, we show that it is not true that members of the smaller group are always more aggressive, even for identical values, when cost distributions differ.

Example 3. \( F_1(c) = c^a \) and \( F_2(c) = c^b \) on \([0,1]\) with \( a > 0 \) and \( b > 0 \), \( n_1 = 1 \), and \( n_2 = 2 \). While now Proposition 3 does not apply, one can still work with Theorem 5 to find the equilibrium. Here, (11) reads as

\[
\xi'(c) = \frac{v_2 c a^{a-1}}{2v_1 \xi(c) b(\xi(c))^{b-1}},
\]

which can be solved as

\[
\xi(c) = \left( \frac{v_2}{2v_1} \frac{a}{a+1} \frac{b+1}{b} \right)^{\frac{1}{a+b+1}} c^{\frac{a}{a+b+1}}.
\]

Therefore, if \( \frac{v_2}{2v_1} \frac{a}{a+1} \frac{b+1}{b} \leq 1 \), then \( \xi(1) < 1 \), so \( g_1 \) has no flat spot and \( g_2 \) does. For the rest of this example we only consider this case, since the other case is dealt with similarly to what was done for the previous example. Now, using (12), (13), and (14), we obtain.

\[
g_1(c) = v_2 \left( \frac{v_2}{2v_1} \frac{a}{a+1} \frac{b+1}{b} \right)^{\frac{1}{a+b+1}} \frac{a(b+1)}{2ab-1+b} \left( 1 - c^{\frac{2ab-1+b}{a+b+1}} \right)
\]

\[
g_2(c) = \begin{cases} 
\frac{v_2}{2v_1} \frac{a}{a+1} \frac{b+1}{b}^{\frac{1}{a+b+1}} \frac{a(b+1)}{2ab-1+b} \left( 1 - \left( \frac{v_2}{2v_1} \frac{a}{a+1} \frac{b+1}{b} \right)^{\frac{2ab-1+b}{a+b+1}} c^{\frac{2ab-1+b}{a+b+1}} \right), & \text{if } c \leq \left( \frac{v_2}{2v_1} \frac{a}{a+1} \frac{b+1}{b} \right)^{\frac{1}{a+b+1}} \\
0, & \text{otherwise.}
\end{cases}
\]

For the sake of concreteness, set \( v_1 = v_2 = 1 \), \( a = 1/2 \), and \( b = 1 \). Then \( g_1(c) = (1 - c^{1/2}) \) and

\[
g_2(c) = \begin{cases} 
1 - (3c^2)^{1/3}, & \text{if } c \leq 1/\sqrt{3} \\
0, & \text{otherwise.}
\end{cases}
\]

In contrast with equilibrium when Proposition 3 applies, the strategies described above cross, with \( g_1(c) \leq g_2(c) \) for \( c \leq 1/9 \).

4.2 Equilibrium strategies with mass at the maximum effort

We next derive jump equilibria analogous to those in Section 3.2 when groups use different strategies, now assuming both groups have at least two members. For each \( \ell = 1, 2 \), let \( g_\ell \) be an equilibrium strategy.
such that all group \( \ell \) types in \([c, c_{0\ell}]\) exert the maximum effort \( \bar{x} \) (to be determined), \( g_\ell \) exhibits a jump discontinuity at some \( c_{0\ell} \in (c, \bar{c}) \) and is continuous on \((c_{0\ell}, \bar{c})\). By Lemma 1, the upper bound \( x' \) of the continuous part of the support should be the same for both groups. Thus \( x' = \lim_{c \to c_{01}^+} g_1(c) = \lim_{c \to c_{02}^+} g_2(c) \) and \( x' < \bar{x} \). As before, let \( c_1^M = \max\{c_{12}, \ldots, c_{1n_1}\} \), so that the relevant distributions for \( c_1^M \) are the cdf \( Q(c_1^M) = [F_1(c_1^M)]^{n_1-1} \) and the density \( q(c_1^M) = (n_1 - 1) [F_1(c_1^M)]^{n_1-2} f_1(c_1^M) \).

On \((c_{01}, \bar{c})\) the equilibrium strategy coincides with that derived in the previous subsection because the (greater) efforts of lower-cost players are not determinative. If all types in \([c, c_{01}]\) exert effort \( \bar{x} \), then for \( n_1 \geq 2 \) these types have no incentive to increase effort above \( \bar{x} \). Moreover, they are willing to exert exactly effort \( \bar{x} \) if the type \( c_{01} \) player is just indifferent between \( \bar{x} \) and \( x' \). Similar reasoning applies to group 2. The relevant interim payoffs for the type-\( c_{01} \) player are

\[
U_1(x'; c_{01}) = v_1 \int_{c_{01}}^{\bar{c}} \Pr\left(g_2(c_{2j}) < g_1(c_1^M), \text{ for some } j = 1, \ldots, n_2\right) h_1(c_1^M) \, dc_1^M
+ v_1 \left(1 - [F_2(c_{02})]^{n_2}\right) [F_1(c_{01})]^{n_1-1} - c_{01}x'
\]

and

\[
U_1(\bar{x}; c_{01}) = v_1 \int_{c_{01}}^{\bar{x}} \Pr\left(g_2(c_{2j}) < g_1(c_1^M), \text{ for some } j = 1, \ldots, n_2\right) h_1(c_1^M) \, dc_1^M
+ v_1 \left(1 - \frac{1}{2} [F_2(c_{02})]^{n_2}\right) [F_1(c_{01})]^{n_1-1} - c_{01}\bar{x},
\]

where the latter equation uses the tie-breaking rule that if all players choose effort \( \bar{x} \), then each group has a 1/2 chance of winning.

Indifference for a type-\( c_{01} \) player in group 1 implies

\[
0 = U_1(x'; c_{01}) - U_1(\bar{x}; c_{01}) = c_{01}(\bar{x} - x') - \frac{v_1}{2} (F_1(c_{01}))^{n_1-1} (F_2(c_{02}))^{n_2}.
\]  \hspace{1cm} (15) \hspace{1cm} \text{asind1}

Similarly, indifference for type \( c_{02} \) in group 2 yields

\[
0 = c_{02}(\bar{x} - x') - \frac{v_2}{2} (F_1(c_{01}))^{n_1} (F_2(c_{02}))^{n_2-1}.
\]  \hspace{1cm} (16) \hspace{1cm} \text{asind2}

Equations (15) and (16) together imply

\[
v_2 c_{01} F_1(c_{01}) = v_1 c_{02} F_2(c_{02}).
\]  \hspace{1cm} (17) \hspace{1cm} \text{ind}
Because the left- (right-) hand side of (17) is strictly increasing in \( c_{01} (c_{02}) \), for a given \( c_{01} \in [\underline{c}, \bar{c}] \) there exists at most one \( c_{02} \) satisfying (17). In particular, when \( c_{01} = \underline{c} \), we have \( c_{02} = \underline{c} \). Consider the following possibilities:

1. \( v_1 < v_2 \). When \( c_{01} = \bar{c} \), there is no \( c_{02} \) that solves (17). Define the function \( \nu_1(c) = cF_1(c) \), which is strictly increasing in \( c \). Then there exists a unique \( \nu_1^{-1}(\frac{v_1\nu}{v_2}) \). It can be verified that when \( c_{01} > \nu_1^{-1}(\frac{v_1\nu}{v_2}) \), (17) never holds. When \( \underline{c} \leq c_{01} \leq \nu_1^{-1}(\frac{v_1\nu}{v_2}) \), there exists a unique \( c_{02} \) that solves (17).

2. \( v_1 = v_2 \). Define \( \nu_2(c) = cF_2(c) \), and using the fact that \( \nu_2 \) is strictly increasing, establish that for every \( c_{01} \in [\underline{c}, \bar{c}] \) there exists a unique \( c_{02} \) that solves (17). The corresponding solution ranges from \( \underline{c} \) to \( \bar{c} \) as \( c_{01} \) ranges from \( \underline{c} \) to \( \bar{c} \). Note that when \( \underline{c} < c_{01} < \bar{c} \), it is generally not true that \( c_{02} = c_{01} \) unless \( F_1 = F_2 \).

3. \( v_1 > v_2 \). Similarly, for every \( c_{01} \in [\underline{c}, \bar{c}] \) there exists a unique \( c_{02} \) that solves (17). The corresponding solution ranges from \( \underline{c} \) to \( \nu_2^{-1}(\frac{v_2\nu}{v_1}) \) as \( c_{01} \) ranges from \( \underline{c} \) to \( \bar{c} \).

Note that, quite interestingly, group size does not enter (17); what matters is the difference in values and cost distributions. Of course, in equilibrium \( g_1 \) could be either left- or right-continuous, with \( g_1(c_{01}) \) equaling either \( x' \) or \( \bar{x} \). The same is true about \( g_2 \). For any \( c \in (c_{01}, \bar{c}] \) the equilibrium strategy \( g_1 \) is given by (12), where \( g_1'(\tau) = 0 \) if \( g_1(\tau) = 0 \), otherwise \( g_1'(\tau) \) is determined by (13), and \( \xi \) is the unique solution to the initial value problem given by (11) together with the initial condition \( \xi(c_{01}) = c_{02} \). Then

\[
x' = \lim_{c \downarrow c_{01}} g_1(c).
\]

Finally, \( \bar{x} \) is determined from (15). We thus have the following result.

**Theorem 6.** Fix \( n_1 \geq 2 \) and \( n_2 \geq 2 \). For each \( i = 1, 2 \), let \( \nu_i(c) = cF_i(c) \). There is a continuum of non-degenerate Bayes-Nash equilibria, indexed by \( c_{01} \in C \subseteq (\underline{c}, \bar{c}] \), with mass at the maximum effort. The range of the mass at the maximum effort in group 1 is as follows.

1. If \( v_1 < v_2 \), then \( C = (\underline{c}, \nu_1^{-1}(\frac{v_1\nu}{v_2})] \).

2. If \( v_1 \geq v_2 \), then \( C = (\underline{c}, \bar{c}) \).

Given any \( c_{01} \in C \), any cost type \( c \in (c_{01}, \bar{c}] \) in group 1 uses strategy \( g_1(c) = -\int_\xi^{\bar{x}} g_1'(\tau) \, d\tau \), with \( g_1'(\tau) = 0 \) if \( g_1(\tau) = 0 \), and otherwise

\[
g_1'(\tau) = -\frac{n_1v_2F_2(\xi(\tau))^{n_2-1}F_1(\tau)^{n_1-1}f_1(\tau)}{\xi(\tau)},
\]

Finally, \( \bar{x} \) is determined from (15). We thus have the following result.
where $\xi$ is the unique solution to the initial value problem given by (11) together with the initial condition $\xi(c_{01}) = c_{02} \equiv \nu_2^{-1}\left(\frac{\nu_1 c_{01} F_1(c_{01})}{F_2(c_{01})}\right)$. Cost type $c_{02} < c \leq \bar{c}$ in group 2 that exerts positive effort uses strategy

$$g_2(c) = g_1\left(\frac{c}{\nu_2}\right).$$

(18)

Group-1 types in $[c, c_{01})$ and group-2 types in $[c, c_{02})$ contribute

$$\bar{x} = \lim_{c \downarrow c_{01}} g_1(c) + \frac{\nu_1}{2c_{01}} \left[F_1(c_{01})\right]^{n_1-1} \left[F_2(c_{02})\right]^{n_2},$$

while group-1 type $c_{01}$ and group-2 type $c_{02}$ are indifferent between contributing $\bar{x}$ and

$$x' = \lim_{c \downarrow c_{01}} g_1(c).$$

(19)

Theorem 6 implies that, when $v_1 < v_2$, the equilibrium mass that group-1 members place on $\bar{x}(c_{01})$ varies between 0 and some intermediate level\(^{10}\) (less than 1), while equilibrium mass in group 2 varies between 0 and 1. Notice that as $c_{01}$ approaches $c$, the equilibrium converges to the one without coordination of Theorem 5, while as $c_{01}$ approaches the highest possible level $\nu_1^{-1}\left(\frac{c_{02}}{v_2}\right)$, the equilibrium converges to the semi-degenerate equilibrium of Theorem 2 corresponding to $k = 2$, where all players in group 2 contribute $\bar{c}$ while players in group 1 contribute 0 or $\bar{c}$ with some positive probabilities. The case $v_1 > v_2$ provides analogous insights.

When $v_1 = v_2$, equilibrium masses on $\bar{x}(c_{01})$ by groups 1 and 2 increase from 0 to 1 as $c_{01}$ increases from $c$ to $\bar{c}$. When $c < c_{01} < \bar{c}$, those two masses are not equal, generally. As $c_{01}$ approaches $\bar{c}$, the equilibrium converges to the degenerate equilibrium where all cost types contribute $\frac{\nu_1}{\nu_2}$. As $c_{01}$ approaches $c$, the equilibrium converges to the one of Theorem 5. Therefore, similarly to Section 3, we can interpret increases in $c_{01}$ and $c_{02}$ as increases in the degree of coordination on effort levels within each group.

The following result is analogous to Proposition 3 and provides sufficient conditions for ordering groups' strategies.

**Proposition 4.** Consider an equilibrium of Theorem 6 with cost cutoffs $c_{01}$ and $c_{02}$. This equilibrium has the following property.

1. If $\frac{F_1(c_{01})}{F_2(c_{01})} > \frac{\nu_1}{\nu_2}$ and $\frac{\nu_1}{\nu_2} < \frac{v_2}{\nu_1} \cdot \frac{f_1(c)}{f_2(c)}$ for all $c \geq c_{01}$, then $g_1(c) < g_2(c) \forall c \in (c_{01}, \bar{c})$. Every player in group 1 puts mass at zero, while the group-2 effort distribution is atomless at zero.

2. If $\frac{F_1(c_{01})}{F_2(c_{01})} = \frac{\nu_1}{\nu_2}$ and $\frac{\nu_1}{\nu_2} = \frac{v_2}{\nu_1} \cdot \frac{f_1(c)}{f_2(c)}$ for all $c \geq c_{01}$, then $g_1(c) = g_2(c) \forall c \in (c_{01}, \bar{c})$ and neither group...
puts mass at zero.

3. If \( \frac{f_1(c_{01})}{f_2(c_{01})} < \frac{v_1}{v_2} \) and \( \frac{v_1}{v_2} > \frac{f_1(c)}{f_2(c)} \) for all \( c \geq c_{01} \), then \( g_1(c) > g_2(c) \) \( \forall c \in (c_{01}, \bar{c}) \). Every player in group 1 uses a strategy that is atomless at zero, while players in group 2 put mass at zero.

One might conjecture that a “stronger” group (in the sense of higher valuation or smaller size) puts greater mass at the upper bound of the support. However, this is not true in general. First, group size does not enter (17). Second, Table 1 below shows that, if \( F_1 \neq F_2 \), then the mass at the top effort level can be largest for the group with the lowest valuation.\(^{11}\)

Table 1: Examples of upper bound mass when \( F_1(c) = c, F_2(c) = c^2 \) on \([0, 1]\); \( v_1 = 1 \), and \( v_2 = 2 \)

<table>
<thead>
<tr>
<th>( c_{01} )</th>
<th>( c_{02} )</th>
<th>( F_1(c_{01}) )</th>
<th>( F_2(c_{02}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>0.27</td>
<td>0.10</td>
<td>0.07</td>
</tr>
<tr>
<td>0.20</td>
<td>0.43</td>
<td>0.20</td>
<td>0.19</td>
</tr>
<tr>
<td>0.30</td>
<td>0.56</td>
<td>0.30</td>
<td>0.32</td>
</tr>
</tbody>
</table>

We conclude this section with a full description of equilibria with mass at the top effort in an example. There are two main “applied” points of interest in the example:

1. The exploration of the degrees of coordination within the two groups, identified with the sizes of the mass point at the top, and so by \( c_{01} \) and \( c_{02} \), that are mutually consistent in equilibrium, and

2. The consequences on the probability of victory of changing these degrees of coordination.

Note in the symmetric equilibria of Section 3 these issues are trivial. The form of equilibrium strategies in Example 4 below is provided for completeness, but, to understand the main applied points of interest, it is sufficient to focus on the form of the function \( \xi \).

**Example 4.** \( F_1 \) and \( F_2 \) are uniform on \([0, 1]\), \( n_1 = 2 \), and \( n_2 = 3 \). Recall that we denote \( \nu_i(c) = cF_i(c) \), so \( \nu_1(c) = c^2 \) and \( \nu_2^{-1}(y) = \sqrt{y} \). According to Theorem 6, the range of mass that players may place at the maximum effort depends on which group has a greater valuation.

**CASE 1:** \( v_1 < v_2 \). To simplify notation, denote \( c_{01} \) by \( r \). According to Theorem 6, \( r \) could range from 0 to \( \sqrt{\frac{v_2}{v_1}} < 1 \). The boundary condition in Theorem 6 becomes

\[
\xi(r) = r\sqrt{\frac{v_2}{v_1}}.
\]

\(^{11}\)Simulations show that first-order stochastic dominance does not provide clear-cut conclusions, either.
Now (11) reads as
\[ \xi(c) \xi'(c) = \frac{2v_2}{3v_1} c, \]
which, together with the aforementioned boundary condition, can be solved as
\[ \xi(c) = \sqrt{\frac{v_2(2c^2 + r^2)}{3v_1}}. \]

Consider the following subcases:

1. \( \frac{3v_1}{2v_2} \geq 1 \), then, depending on \( r \), either group could be the first to drop out.
   
   (a) If \( 0 \leq r \leq \sqrt{\frac{3v_1}{v_2} - 2} \) (one can verify that when \( v_1 < v_2 \), \( \sqrt{\frac{3v_1}{v_2} - 2} < \sqrt{\frac{v_1}{v_2}} \)), then group 2 drops out before group 1, and \( g_1 \) is atomless on \((r, 1]\). Solving for equilibrium bidding rules using (12), (13), and (14), we obtain:

   \[
g_1(c) = \begin{cases} \frac{v_1^{3/2}}{3\sqrt{3v_1}} \left( 2 + r^2 \right)^{3/2} - \frac{v_2^{3/2}}{2\sqrt{v_1}} r^3 & \text{if } 0 \leq c < r \\ \left\{ \frac{v_1^{3/2}}{3\sqrt{3v_1}} \left( 2 + r^2 \right)^{3/2} - \frac{v_2^{3/2}}{2\sqrt{v_1}} r^3, \frac{v_1^{3/2}}{3\sqrt{3v_1}} \left( 2 + r^2 \right)^{3/2} - \frac{v_2^{3/2}}{2\sqrt{v_1}} r^3 \right\} & \text{if } c = r \\ \frac{v_1^{3/2}}{3\sqrt{3v_1}} \left( 2 + r^2 \right)^{3/2} - \left( 2c^2 + r^2 \right)^{3/2} & \text{if } r < c \leq 1 \end{cases} \]

   \[
g_2(c) = \begin{cases} \frac{v_1^{3/2}}{3\sqrt{3v_1}} \left( 2 + r^2 \right)^{3/2} - \frac{v_2^{3/2}}{2\sqrt{v_1}} r^3 & \text{if } 0 \leq c < r \sqrt{\frac{2c}{v_1}} \\ \left\{ \frac{v_1^{3/2}}{3\sqrt{3v_1}} \left( 2 + r^2 \right)^{3/2} - \frac{v_2^{3/2}}{2\sqrt{v_1}} r^3, \frac{v_1^{3/2}}{3\sqrt{3v_1}} \left( 2 + r^2 \right)^{3/2} - \frac{v_2^{3/2}}{2\sqrt{v_1}} r^3 \right\} & \text{if } c = r \sqrt{\frac{2c}{v_1}} \\ \frac{v_1^{3/2}}{3\sqrt{3v_1}} \left( 2 + r^2 \right)^{3/2} - v_1 c^3 & \text{if } r \sqrt{\frac{2c}{v_1}} < c \leq \sqrt{\frac{v_2(2 + r^2)}{3v_1}} \\ 0 & \text{if } c > \sqrt{\frac{v_2(2 + r^2)}{3v_1}} \end{cases} \]

   (b) If \( r > \sqrt{\frac{3v_1}{v_2} - 2} \), then group 1 drops out before group 2, and cost types at or above \( \sqrt{\frac{3v_1}{2v_2} - \frac{r^2}{2}} \) in group 1 contribute zero. One can solve for equilibrium bidding strategies as follows:

   \[
g_1(c) = \begin{cases} v_1 - \frac{v_1^{3/2}}{2\sqrt{v_1}} r^3 & \text{if } 0 \leq c < r \\ \left\{ v_1 - \frac{v_1^{3/2}}{2\sqrt{v_1}} r^3, v_1 - \frac{v_1^{3/2}}{2\sqrt{v_1}} r^3 \right\} & \text{if } c = r \\ v_1 - \frac{v_1^{3/2}}{3\sqrt{3v_1}} \left( 2c^2 + r^2 \right)^{3/2} & \text{if } \sqrt{\frac{3v_1}{2v_2} - \frac{r^2}{2}} < c < r \\ 0 & \text{if } c > \sqrt{\frac{3v_1}{2v_2} - \frac{r^2}{2}} \end{cases} \]
\[ g_2(c) = \begin{cases} 
  v_1 - \frac{v_1^{3/2} r^3}{2v_1 v_2} & \text{if } 0 \leq c < r \sqrt{\frac{v_2}{v_1}} \\
  v_1 - \frac{v_1^{3/2}}{v_1^2} r^3, v_1 - \frac{v_1^{3/2} r^3}{2v_1 v_2} & \text{if } c = r \sqrt{\frac{v_2}{v_1}} \\
  v_1 (1 - c^3) & \text{if } r \sqrt{\frac{v_2}{v_1}} < c \leq 1.
\end{cases} \]

2. \( \frac{3v_1}{2v_2} \leq 1 \), then group 1 drops out before group 2, regardless of \( r \). This case could be worked out analogously to Case 1.1.b.

**CASE 2:** \( v_1 \geq v_2 \). One can work out this case analogously and find the same expression for \( \xi \),

\[ \xi(c) = \sqrt{\frac{v_2 (2c^2 + r^2)}{3v_1}}. \]

However, the range of mass is different, as \( r \) could vary between 0 and 1, while \( \xi(r) \) – between 0 and \( \sqrt{\frac{v_2}{v_1}} < 1 \). In this case, \( \frac{3v_1}{2v_2} > 1 \), and group 2 drops out before group 1 regardless of \( r \). This case is analogous to Case 1.1.a.

We now turn to the applied points of the example, noting that changes in \( r \) have important consequences. First, we see that increases in coordination in group 1, i.e., increases in \( c_{01} = r \), are only consistent in equilibrium with increases in coordination in group 2, i.e., with increases in \( c_{02} = \xi(r) \). This point holds generally, because the left- (right-) hand side of (17) is strictly increasing in \( c_{01} (c_{02}) \). But the exact form depends on values and distributions. Second, we now calculate how the probability of victory of group 1 is affected by an increase in coordination, measured by \( r \). For clarity, we set \( v_1 = v_2 \), so that group 1 is unambiguously the stronger group, as cost distributions are identical and \( n_1 < n_2 \). Now, the initial condition for \( \xi \) is \( \xi(r) = r \), i.e., \( c_{01} = c_{02} = r \), so

\[
\Pr\{ \text{Group 1 wins} \} = \int_0^{c_{01}} \left[ \int_0^{c_{02}} \frac{1}{2} d(c_2)^3 + \int_{c_{02}}^r d(c_2)^3 \right] d(c_1)^2 + \int_{c_{01}}^1 \left[ \int_{\xi(c_1)}^1 d(c_2)^3 \right] d(c_1)^2 \\
\int_0^r \left[ \int_0^{c_{02}} \frac{1}{2} d(c_2)^3 + \int_{c_{02}}^r d(c_2)^3 \right] d(c_1)^2 + \int_0^1 \left[ \int_{\sqrt{\frac{2c_1^2 + r^2}{4}}}^{c_1} d(c_2)^3 \right] d(c_1)^2 \\
= \frac{90 + 9r^5 - 2\sqrt{3}(2 + r^2)^{5/2}}{90}.
\]

Note that this is a strictly decreasing function of \( r \). Therefore, increases in coordination in the two groups end up eroding the advantage of the stronger group 1.
Cheap-talk communication within groups

In this section, we consider the effects of group members coordinating their efforts within each group via cheap-talk communication. The complementarity of efforts of the weakest-link technology offers strong rewards to coordination, since wasted contributions are eliminated and agents are willing to contribute more, because they do not fear that teammates will not match their effort. As demonstrated in Barbieri and Malueg (2018), it is possible to design simple and intuitive schemes that provide agents with the appropriate individual incentives to coordinate their behavior in a weakest-link public good model.

Here, we focus on one such scheme: before jointly deciding on a contribution level, teammates exchange information about their cost realization and then they all contribute the effort level most preferred by the agent with the highest cost. In their public good model, Barbieri and Malueg (2018) show that truth-telling is an equilibrium and all agents benefit. In our symmetric setup with 2 groups of \( n \) agents each, it turns out that a similar result goes through if one group coordinates its efforts as described above. In the rest of this section, we focus on equilibria without mass points at the maximum effort.\(^{12}\)

**Proposition 5.** Consider a symmetric environment with 2 groups of \( n \) agents each. Assume that within group 1 agents coordinate on the effort most preferred by the agent with the largest cost, while agents in group 2 behave as in Sections 3 and 4, i.e., they do not coordinate. With respect to the situation in which neither group coordinates described in Theorem 3, the probability of victory of group 1 increases and the interim expected utility of any agent in group 1 strictly increases, for any cost type strictly larger than \( c_2 \).

The common-sense result in Proposition 5 leaves open an important question in our contest setup: since the reduction of free-riding within teams is expected to induce harsher competition across teams, do agents actually benefit if they all coordinate their efforts in their respective groups via cheap talk? We again answer this question in our symmetric setup with 2 groups of \( n \) agents each.

First, we determine the equilibrium effort level most preferred by the agent with the largest cost realization \( c^M \). Denote the effort level that all teammates provide after exchanging information in a symmetric equilibrium as \( g_M(c^M) \). If the highest cost agent in group 1 contributes \( x \), in equilibrium she receives utility

\[
v(1 - F(g_M^{-1}(x))^n) - c^M x;
\]

so the FOC yields

\[
-vnF(g_M^{-1}(x))^{n-1} f(g_M^{-1}(x)) \frac{1}{g'_M(g_M^{-1}(x))} = c^M.
\]

\(^{12}\)To avoid unnecessary technical complications, we further assume the density \( f \) is strictly positive on \([c, \bar{c}]\), not just \((c, \bar{c})\) as in previous sections.
In a symmetric equilibrium the above can be evaluated at \( x = g_M(c^M) \), so

\[
g'_M(c^M) = -\frac{vnF(c^M)^{n-1}f(c^M)}{c^M}. \tag{20}
\]

Noting that \( g_M(\bar{c}) = 0 \), the above displayed equation then yields this analog of (2):

\[
g_M(c^M) = vn \int_{c^M}^{\bar{c}} \frac{F(\tau)^{n-1}f(\tau)}{\tau} d\tau. \tag{21}
\]

As expected, the comparison of (2) and (21) reveals the highest-cost agent is more aggressive in the presence of coordination via cheap talk, as \( g_M(c) > g(c) \ \forall \ c \in [\underline{c}, \bar{c}] \), and so each team’s effort is larger for any realization of the costs of its members.

Second, we now show that agents have no incentive to lie when they report their cost realization to their teammates. To see this, consider the utility of a group-1 agent with realized cost \( c \) that is assumed to induce cost \( c^M = z \) in her group:

\[
V_M(z, c) = v(1 - F(z)^n) - cg_M(z).
\]

We then have

\[
\frac{\partial V_M(z, c)}{\partial z} = -\frac{vnF(z)^{n-1}f(z)}{z} z - cg'_M(z)
\]

\[
= \begin{cases} 
  g'_M(z)(z - c); & \text{by (20)} \\
  < 0 & \text{for } z < c;
\end{cases}
\]

therefore, \( V_M(z, c) \) is strictly increasing in \( z \) for \( z < c \) and strictly decreasing for \( z > c \). This implies that any lie when reporting one’s true cost to teammates can only damage this agent. Indeed, if this agent’s cost realization is maximal in the group, then \( z = c \) is optimal by the above reasoning. And if the maximum cost \( c' > c \) belongs to another agent, then a lie can only move \( z \) further to the right of \( c' \). But this is not profitable since \( V_M(z, c) \) decreases in \( z \) for \( z > c \).

We can now compare payoffs in the equilibrium described above, where agents contribute \( g_M(c^M) \) described in (21), and in the equilibrium of Theorem 3. Since in any symmetric equilibrium, each group’s \( \text{ex ante} \) payoff gross of costs is \( v/2 \), we can focus on the comparison of expected costs. Using (21), an agent’s expected cost in the game with cheap talk equals

\[
\int_{\underline{c}}^{\bar{c}} c \left[ \int_{\underline{c}}^{c} g_M(c)dF(\tau)^{n-1} + \int_{c}^{\bar{c}} g_M(\tau)dF(\tau)^{n-1} \right] f(c)dc.
\]
\[ \int_{c}^{\bar{c}} \left[ g_M(c) F(c)^{n-1} + g_M(\bar{c}) F(\bar{c})^{n-1} - g_M(c) F(c)^{n-1} - \int_{c}^{\bar{c}} F(\tau)^{n-1} g'_M(\tau) d\tau \right] f(c) dc \]

\[ = \int_{c}^{\bar{c}} \left[ vn \int_{c}^{\bar{c}} \frac{F(\tau)^{2(n-1)} f(\tau)}{\tau} d\tau \right] f(c) dc, \]

where the first equality follows after integration by parts and the second by (21). But the above displayed equation is the expected cost in the equilibrium described in Theorem 3; indeed, the term in square brackets above is \( g(c) \) in (2). In other words, all within-team gains are lost to increased competition between teams:

**Proposition 6.** In a symmetric environment, agents’ ex ante utility is the same in the symmetric equilibrium with cooperation with the effort strategy described in (21) and in the symmetric equilibrium without cooperation described in Theorem 3.

### 6 Conclusion

This paper contributes to the early but growing literature on group contests with incomplete information and is the first attempt to study players’ incentives to share information in a group contest. We analyzed the interplay between the weakest-link effort aggregation and private information about cost of effort. A remarkable finding is that players within a group always choose symmetric strategies. This contrasts with the literature that assumes alternative effort aggregation technologies, such as “best shot” or additive; that literature documents existence of a champion within each group who exerts a positive level of effort, as well as free riders who contribute zero.

The literature on weakest link contests (Chowdhury et al., 2016) establishes existence of pure strategy equilibria (in which the support of a player’s strategy is a singleton) in a setting where the valuations and the cost of effort are common knowledge. In our setting, where each player is privately informed about her own cost of effort but is unsure about the other players’ costs, a player’s level of effort depends on the private realization of her cost. Therefore, dispersion in the cost of effort may yield dispersion of the effort levels, so that the support of the effort distribution can be a compact interval. For this kind of equilibrium and in a symmetric setup, we show that the effect of symmetrically increasing group size on ex-ante welfare is ambiguous. An increase in the group size exacerbates the free-riding problem, however, players may benefit since the contest becomes less competitive. For a large class of cost distributions (namely, power distributions), the two countervailing effects cancel out so that changes in the group size are welfare-neutral. We characterize the condition under which an increase in the group size increases (decreases) the ex-ante expected payoff.
Interestingly, we show that, even when the cost of effort is a private information, it is still possible for the support of a player’s strategy to be a singleton. In such a situation, all players coordinate on an effort that the highest-cost player is willing to exert. Although other types would be willing to exert greater effort, the weakest-link technology makes such deviations unprofitable.

It is also possible that the support of the effort distribution consists of an interval plus an isolated point. In this case, all players with sufficiently low realized cost contribute the same amount, which is discretely greater than the effort level chosen by any other cost type. The effort distribution then has a mass at the top. One may hypothesize that such coordination of the more productive types is welfare-enhancing, as it partially eliminates the waste of effort that comes as a consequence of the weakest-link effort technology when efforts are dispersed. However, the effect of the size of the mass at the top on ex-ante welfare is, once again, ambiguous. The intuition is that, although coordination helps reduce the waste of effort, higher cost types among those that do coordinate experience a discrete jump in their contribution level, while lower cost types contribute less than what they would in the absence of coordination.\textsuperscript{13} As a limiting case, even all players coordinating on the same effort level may not improve the ex-ante welfare. Interestingly, the same condition that characterizes the effect of the group size on welfare also characterizes the welfare effect of the size of the mass at the top.

It is also interesting to investigate the implications of asymmetries between groups. If groups are symmetric except for group size, players in the smaller group bid more aggressively than those in the larger group, supporting the classical group-size paradox that is attributed to Olson (1965). Surprisingly, this result is analogous to the finding of Barbieri et al. (2014), who find that, in the semi-symmetric equilibria of the group \textit{best-shot} all-pay auction with \textit{complete} information and symmetric valuations, players in the larger group put mass at zero.

When asymmetry between the two groups is along multiple dimensions, such as group size, valuation and cost distribution, the interplay between those three factors determines which group bids more aggressively. Just as in the case of symmetric groups, it is possible that the effort distributions for both groups possess a mass at the top. Remarkably, the relative size of the masses is independent of the group sizes. One might suspect that a “stronger” group (in the sense of higher valuation or smaller size) puts greater mass at the top. This, however, is not true in general. Simulations show that first-order stochastic dominance does not provide clear-cut insights either.

To investigate the role of information exchange, we consider the case where players within a group may exchange information about their cost realization (using cheap talk), and all coordinate on the effort most

\textsuperscript{13}This, of course, happens because all types that coordinate do so at the level of effort that the highest-cost type among them is willing to exert.
preferred by the agent with the largest cost. We show that a player is always better off reporting truthfully. Given that players in the rival group do not coordinate, a group enjoys a greater probability of victory from coordination, compared to the benchmark case of no coordination, and almost every cost type has a higher interim expected utility. This is, of course, due to the mitigation of free-riding within the coordinating group.

A striking result is that, when players within each group coordinate, all within-group gains are lost to increased competition between groups, so that the ex-ante utility is the same in the symmetric equilibria with and without cooperation.

Appendix

Proof of Theorem 2. First, we build a semi-degenerate equilibrium in which group-1 members use a degenerate strategy. Then, we will show that semi-degenerate equilibria in which group-2 members use a degenerate strategy do not exist.

Let all group-1 members bid $\bar{x}_1 > 0$ with probability 1. Then, it must be that $\bar{x}_1 = \bar{x}_2 = \bar{x}$ in equilibrium. If instead $\bar{x}_1 > \bar{x}_2$, then for all cost levels, each player in group 1 could improve her payoff by reducing her bid to $(\bar{x}_1 + \bar{x}_2)/2$, contradicting the assumption of equilibrium. Similarly, if $\bar{x}_1 < \bar{x}_2$, then a positive mass of group 2 types bid in the interval $((\bar{x}_1 + \bar{x}_2)/2, \bar{x}_2]$, and all of these types could increase their interim payoffs by reducing their bids to $(\bar{x}_1 + \bar{x}_2)/2$. The assumption of semi-degenerate supports implies $\bar{x}_1 = \bar{x}$ and $\bar{x}_2 < \bar{x}$. For group-2 members, any bid in $(0, \bar{x})$ is sure to lose and so yield a negative interim payoff. Therefore, any types of group-2 players not bidding $\bar{x}$ must bid 0. Thus, $\bar{x}_2 = 0$ and, by the property of the weakest-link aggregator, each player $i$ in group 2 must put mass $a_i$ at $\bar{x}$, with $0 < a_i \leq 1$, for otherwise group-1 agents would have a profitable downward deviation. Therefore, for any $i$, type $F_{2}^{-1}(a_i)$ must obtain a non-negative payoff: $(v_2/2) \prod_{j \in I_2, j \neq i} a_j - \bar{x}F_2^{-1}(a_i) \geq 0$, or, multiplying both sides by $a_i$,

$$\frac{v_2}{2} \prod_{j \in I_2} a_j \geq \bar{x}a_i F_2^{-1}(a_i), \quad \forall i \in I_2.$$  

Since we are looking for a semi-degenerate equilibrium, there exists at least one agent $i'$ such that $a_{i'} < 1$, so type $F_{2}^{-1}(a_{i'}) \in (\bar{c}, \bar{c})$ is indifferent between the bids of zero and $\bar{x}$ and we have

$$\frac{v_2}{2} \prod_{j \in I_2} a_j = \bar{x}a_{i'} F_2^{-1}(a_{i'}).$$  \hspace{1cm} (22) \hspace{1cm} \text{indif}$$

But the above two displayed equations imply $a_i \leq a_{i'} < 1$ for all $i \in I_2$, because $xF_2^{-1}(x)$ is a strictly increasing function of $x$ and the left-hand sides are the same. But then (22) applies to $i$ as well; hence,
\(a_i = a_{i'} = a\) for all \(i \in I_2\). Equation (22) now implies that

\[\bar{x} = \frac{v_2 a^{n_2 - 1}}{2F_2^{-1}(a)}\]

The utility of player \(i\) in group 1 who contributes \(\bar{x}\) when her cost is \(\bar{c}\) equals

\[U_{1,i}(\bar{x}, \bar{c}) = v_1(1 - a^{n_2}) - \bar{x} \bar{c} = v_1(1 - \frac{1}{2} a^{n_2}) - \frac{v_2}{2} a^{n_2 - 1} \frac{\bar{c}}{F_2^{-1}(a)},\]

while a deviation to an arbitrarily small but positive effort generates utility that approaches

\[U_{1,i}(0^+, c) = v_1(1 - a^{n_2}),\]

as the contribution approaches 0. Therefore, a cost type \(\bar{c}\) in group 1 will not deviate to a contribution lower than \(\bar{x}\) if and only if \(U_{1,i}(\bar{x}, \bar{c}) \geq U_{1,i}(0^+, c)\), or

\[aF_2^{-1}(a) \geq \frac{v_2}{v_1} \bar{c}.\]  \hspace{1cm} (23)

Note that \(aF_2^{-1}(a)\) is strictly increasing in \(a\) with a maximum at \(\bar{c}\) for \(a = 1\). When \(v_1 > v_2\), there exists a unique \(a^* < \bar{c}\) where

\[a^*F_2^{-1}(a^*) = \frac{v_2}{v_1} \bar{c},\]  \hspace{1cm} (24)

such that (23) is satisfied if and only if \(a \geq a^*\). The unique \(a^*\) determined by (24) is the lower bound on the mass that each player in group 2 places at \(\bar{x}\). If \(n_1\) and \(n_2\) are both greater than 1, then the above discussion exhausts all possible deviations, since upwards deviations from \(\bar{x}\) are not profitable by the properties of the weakest-link aggregator. This concludes the proof that the strategy described in the statement of the theorem is an equilibrium.

We now turn to the necessary conditions in the statement of the theorem. First, note that if at least one group is composed of only one agent, then this agent has a strictly profitable upwards marginal deviation from \(\bar{x}\), and no semi-degenerate equilibrium can exist. Second, when \(v_2 \geq v_1\), (23) is never satisfied for the admissible range of \(a\) for a semi-degenerate equilibrium. This implies that semi-degenerate equilibria do not exist when \(v_2 = v_1\) and, if they exist for \(v_2 \neq v_1\), then the degenerate strategy must be adopted by the members of the group with the larger valuation.

**Proof of Lemma 1.** 1. Suppose \(\bar{x}_1 > \bar{x}_2\). Then, by the properties of the weakest-link aggregator, all players in group 1 bid in \((\frac{\bar{x}_1 + \bar{x}_2}{2}, \bar{x}_1]\) with strictly positive probability, so each is better off by reducing such bids to
and we have a contradiction to equilibrium. Therefore $\bar{x}_1 = \bar{x}_2 = \bar{x}$. By definition of non-degenerate equilibrium, $\bar{x} > \bar{x}_l$, $l = 1, 2$. In the rest of the proof, many arguments rest on agents increasing their probability of success through an increase in their bid. Since $\bar{x}_1 = \bar{x}_2 = \bar{x}$, the properties of the weakest-link aggregator imply that all agents contribute with strictly positive probability amounts arbitrarily close to $\bar{x}$. Therefore, all agents contributing any amount strictly lower than $\bar{x}$ can increase the probability of winning by increasing their efforts.

Next suppose $\bar{x}_1 > \bar{x}_2$. Then no agent in group 2 bids in $(0, \bar{x}_1)$ because such bids are costly but yield zero chance of winning; therefore, $0 = \bar{x}_2$ and $H_2(0) > 0$. Furthermore, there must be a positive mass for $H_2$ in $[\bar{x}_1, \bar{x}_1 + \varepsilon]$, for any $\varepsilon > 0$, otherwise one of the group 1 agents contributing at or near $\bar{x}_1$ could profitably reduce her contribution. If there is a mass point for $H_2$ at $\bar{x}_1$, then there must be a mass point for $H_1$ at $\bar{x}_1$, for otherwise a group 2 agent contributing $\bar{x}_1$ with positive probability could profitably lower her contribution to zero. But if $H_1$ and $H_2$ have mass points at $\bar{x}_1$, then a group 2 agent contributing $\bar{x}_1$ with positive probability could increase her payoff by increasing this bid to $\bar{x}_1 + \delta$, for $\delta > 0$ sufficiently small, and we reach a contradiction. And if, instead, $H_2$ has no mass point at $\bar{x}_1$, then, for $\delta > 0$ sufficiently small, there exists a strictly positive mass of group 1 agents who are contributing between $\bar{x}_1$ and $\bar{x}_1 + \delta$ who, by reducing their contribution to $\varepsilon$, can obtain a discrete savings in cost while sustaining an infinitesimal drop in the probability of winning, thus raising their payoffs. This too is a contradiction, so it must be that $\bar{x}_1 = \bar{x}_2 = \bar{x}$.

We now show $\bar{x} = 0$, proceeding again by contradiction. If $\bar{x} > 0$ and both groups have a mass point at $\bar{x}$, then there is a profitable upwards deviation to $\bar{x} + \varepsilon$, for $\varepsilon > 0$ sufficiently small. And if group 1, say, does not have a mass point at $\bar{x}$, then agents in group 2 bidding sufficiently close to $\bar{x}$ can reduce their bids to zero and save a discrete amount on their bidding costs while suffering an infinitesimal decrease in their probability of winning. In either case, a contradiction arises so $\bar{x}$ must be zero.

2. By contradiction, suppose that $H_2$ has a mass point at $\gamma \in (0, \bar{x})$. Then no agent in group 1 should contribute with strictly positive probability in $(\gamma - \varepsilon, \gamma]$, for $\varepsilon > 0$ sufficiently small. To see this, note that all agents in group 1 are bidding in a neighborhood of $\bar{x}$ with strictly positive probability. So if an agent raises her bid from anything in $(\gamma - \varepsilon, \gamma]$ to $\gamma + \delta$, her probability of winning increases discontinuously while, for $\delta > 0$ sufficiently small, cost increases only marginally. Having thus established that no contributions in group 1 fall in $(\gamma - \varepsilon, \gamma]$ with strictly positive probability, we see agents in group 2 bidding $\gamma$ have a strictly profitable deviation to $\gamma - \varepsilon/2$: they save on the cost with no repercussions on the probability of winning. To see that at most one group puts mass at zero, note that if both groups had a mass point at 0, then there is a profitable upwards deviation to $\varepsilon$, for $\varepsilon > 0$ sufficiently small.

3. Denote the support of $H_i$ with $S_i$, $i = 1, 2$. Let $t \in S_1$ be such that $t \notin S_2$. Then there exists a
non-empty interval \((a, b) \subset [0, \bar{x}]\) such that \(t \in (a, b)\), \((a, b) \subseteq S_1\) by Part 2 above, and \((a, b) \cap S_2 = \emptyset\). Therefore, group 1 agents bidding in \((\frac{a+b}{2}, b)\) have a strictly profitable deviation to \(\frac{a+b}{2}\), since this does not change the probability of victory but reduces cost.

4. Suppose to the contrary that there exists an interval \((a, b)\) with \(0 < a < b < \bar{x}\) such that \((a, b) \cap S = \emptyset\), but \([b, b + \varepsilon) \subset S\), for some \(\varepsilon > 0\). Then agents bidding in \([b, b + \varepsilon)\) have a strictly profitable deviation to \(\frac{a+b}{2}\) unless a mass point for \(H_1\) and \(H_2\) exists at \(b\). But, by Part 2, that is possible only if \(b = \bar{x}\). Suppose now that \(H_1\) admits a mass point at \(\bar{x}\). Then, for \(\varepsilon > 0\) sufficiently small, no contribution in group 2 can fall in \((\bar{x} - \varepsilon, \bar{x})\), for otherwise a profitable deviation to \(\bar{x}\) results. But since \(\bar{x}\) must remain in \(S_2\), then \(H_2\) has a mass point at \(\bar{x}\). And since \(\bar{x} - \varepsilon < \bar{x}\), then there exists \(a < \bar{x}\) such that neither \(H_1\) nor \(H_2\) puts positive probability on \((a, \bar{x})\). Suppose now that \(H_1\) does not admit a mass point at \(\bar{x}\). Then there is a strictly positive probability of group 1’s contributions falling in \((\bar{x} - \varepsilon, \bar{x})\), for any \(\varepsilon > 0\), which implies there cannot be a mass point for \(H_2\) at \(\bar{x}\). Therefore, \(a = \bar{x}\), i.e., \(S = [0, \bar{x}]\). \(\square\)

**Proof of Lemma 2.** We focus on members of group 1; considerations for group 2 follow similarly. With a contribution of \(\gamma < \bar{x}\), player \(i\) with cost \(c\) obtains equilibrium utility

\[
U_i(\gamma; c) = v(1 - H_1^{-i}(\gamma))H_2(\gamma) + v \int_0^\gamma H_2(s) dH_1^{-i}(s) - c\gamma, \tag{25}\]

a continuous function of both \(\gamma\) and \(c\). Integration by parts yields

\[
U_i(\gamma; c) = v \int_0^\gamma (1 - H_1^{-i}(s)) dH_2(s) - c\gamma + vH_2(0)(1 - H_1^{-i}(0)), \tag{26}\]

so that

\[
\frac{\partial}{\partial \gamma} U_i(\gamma; c) = v(1 - H_1^{-i}(\gamma)) dH_2(\gamma) - c. \tag{27}\]

Furthermore, note that for any two teammates \(i\) and \(j\), \(i \neq j\), we have

\[
1 - H_1^{-i}(\gamma) = (1 - H_1^{-i,j}(\gamma))(1 - H_1^j(\gamma)). \tag{28}\]

We complete the proof through a series of steps. The following step shows that, essentially, equilibrium strategies can have no jumps, except possibly at \(\bar{x}\).

**Step 1.** Consider \(0 \leq \gamma_l < \gamma_h < \bar{x}\). \(H_1^l\) cannot put zero probability over \((\gamma_l, \gamma_h)\) while at the same time putting strictly positive probability on \((\gamma_l - \varepsilon, \gamma_l]\) and \([\gamma_h, \gamma_h + \varepsilon)\) for \(\varepsilon > 0\) sufficiently small.

**Proof.** We proceed by contradiction. If \(H_1^l\) puts zero probability over \((\gamma_l, \gamma_h)\), then, by continuity of \(U_i\),
a necessary condition for equilibrium is that for some \( \hat{c}_i \) we have \( U_i(\gamma; \hat{c}_i) = U_i(\gamma_h; \hat{c}_i) \), and

\[
\lim_{\gamma \downarrow \gamma} \frac{\partial}{\partial \gamma} U_i(\gamma; \hat{c}_i) \leq 0 \leq \lim_{\gamma \uparrow \gamma_h} \frac{\partial}{\partial \gamma} U_i(\gamma; \hat{c}_i). \tag{29} \]

By Lemma 1, the support of \( H_1 \) includes \( (\gamma_l, \gamma_h) \); therefore, there must exist one other agent in group one, \( j \), say, that bids on a dense subset of \( (\gamma_l, \gamma_l + \varepsilon) \). In this interval, \( j \)'s FOC, using (27) and (28), reads as

\[
g_{i,j}^{-1}(\gamma) = v(1 - H_1^{-i,j}(\gamma))(1 - H_1^j(\gamma)) dH_2(\gamma)
= v(1 - H_1^{-i,j}(\gamma))F_1(\hat{c}_i) dH_2(\gamma). \tag{30} \]

Similarly, some group one agent \( k \) bids in \( (\gamma_h - \varepsilon, \gamma_h) \), with strategy \( g_{i,k} \) satisfying the FOC

\[
g_{i,k}^{-1}(\gamma) = v(1 - H_1^{-i,k}(\gamma))F_1(\hat{c}_i) dH_2(\gamma). \tag{31} \]

Now suppose player \( i \) with cost \( \hat{c}_i \) deviates to \( \gamma \in (\gamma_l, \gamma_l + \varepsilon) \). In this range, (27), (28), and (30) imply

\[
\frac{\partial}{\partial \gamma} U_i(\gamma; \hat{c}_i) = v(1 - H_1^{-i,j}(\gamma))(1 - H_1^j(\gamma)) dH_2(\gamma) - \hat{c}_i \quad \text{by (27) and (28)}
= \frac{g_{i,j}^{-1}(\gamma)}{F_1(\hat{c}_i)} (1 - H_1^j(\gamma)) - \hat{c}_i \quad \text{by (30)}
= \frac{g_{i,j}^{-1}(\gamma)}{F_1(\hat{c}_i)} F_1(g_{i,j}^{-1}(\gamma)) - \hat{c}_i.
\]

Define \( \hat{c}_j \equiv \lim_{\gamma \downarrow \gamma} g_{i,j}^{-1}(\gamma) \). Now the first limit in (29) yields

\[
0 \geq \lim_{\gamma \downarrow \gamma} \frac{\partial}{\partial \gamma} U_i(\gamma; \hat{c}_i) = \lim_{\gamma \downarrow \gamma} \frac{g_{i,j}^{-1}(\gamma)}{F_1(\hat{c}_i)} F_1(g_{i,j}^{-1}(\gamma)) - \hat{c}_i = \frac{1}{F_1(\hat{c}_j)} (\hat{c}_j F_1(\hat{c}_j) - \hat{c}_j F_1(\hat{c}_i)).
\]

Consequently, because \( cF_1(\varepsilon) \) is strictly increasing at \( \hat{c}_i \), it follows that \( \hat{c}_j \leq \hat{c}_i \). An analogous argument for player \( k \) yields \( \hat{c}_k \equiv \lim_{\gamma \uparrow \gamma_h} g_{i,k}^{-1}(\gamma) \geq \hat{c}_i \). Therefore,

\[
\hat{c}_j \leq \hat{c}_k. \tag{32} \]

If \( j = k \) then we have a contradiction, because \( g_{i,j} \) is strictly decreasing when taking values in \( (\gamma_l, \gamma_l + \varepsilon) \), weakly decreasing otherwise, and \( \gamma_l < \gamma_h \). Therefore, \( j \neq k \). But we know \( g_{i,j} \) is strictly decreasing when taking values in \( (\gamma_l, \gamma_l + \varepsilon) \) and \( g_{i,k} \) is strictly decreasing when taking values in \( (\gamma_h - \varepsilon, \gamma_h) \). Therefore,
(32) implies

\[ H^j_1(\gamma) > H^k_1(\gamma), \quad \forall \gamma \in (\gamma_l, \gamma_h). \]  

To conclude the proof, note that equilibrium requires \( U_j(\gamma_i; \hat{c}_j) \geq U_j(\gamma_h; \hat{c}_j) \), and \( U_k(\gamma_h; \hat{c}_k) \geq U_k(\gamma_l; \hat{c}_k) \).

Using (26), we have

\[ 0 \leq U_j(\gamma_l; \hat{c}_j) - U_j(\gamma_h; \hat{c}_j) = -v \int_{\gamma_l}^{\gamma_h} (1 - H_1^{-\{j\}}(s)) \, dH_2(s) + \hat{c}_j(\gamma_h - \gamma_l), \]

implying

\[ v \int_{\gamma_l}^{\gamma_h} (1 - H_1^{-\{j\}}(s)) \, dH_2(s) \leq \hat{c}_j(\gamma_h - \gamma_l). \]  

Similarly,

\[ 0 \leq U_j(\gamma_h; \hat{c}_k) - U_j(\gamma_l; \hat{c}_k) = v \int_{\gamma_l}^{\gamma_h} (1 - H_1^{-\{k\}}(s)) \, dH_2(s) - \hat{c}_k(\gamma_h - \gamma_l), \]

implying

\[ \hat{c}_k(\gamma_h - \gamma_l) \leq v \int_{\gamma_l}^{\gamma_h} (1 - H_1^{-\{k\}}(s)) \, dH_2(s). \]  

Because \( \hat{c}_j \leq \hat{c}_k \), (34) and (35) imply

\[ \int_{\gamma_l}^{\gamma_h} (1 - H_1^{-\{j\}}(s)) \, dH_2(s) \leq \int_{\gamma_l}^{\gamma_h} (1 - H_1^{-\{k\}}(s)) \, dH_2(s), \]

or, using (28),

\[ \int_{\gamma_l}^{\gamma_h} (1 - H_1^{-\{j,k\}}(s))(H_1^j(s) - H_1^k(s)) \, dH_2(s) \leq 0, \]

which is impossible by (33).

We now show that if one equilibrium strategy has a jump from \( \gamma_l \) to \( \bar{x} \), then all strategies have the same jump point.

**Step 2.** Consider \( 0 \leq \gamma_l < \gamma_h < \bar{x} \). \( H^j_1 \) cannot put zero probability over \( (\gamma_l, \bar{x}) \), while at the same time putting strictly positive probability on \( (\gamma_l - \varepsilon, \gamma_l) \), for \( \varepsilon > 0 \) sufficiently small, if \( H_1 \) puts strictly positive probability on \( (\gamma_l, \gamma_h) \).

**Proof.** First note that, for any \( \varepsilon > 0 \), all agents put strictly positive probability on \( (\bar{x} - \varepsilon, \bar{x}) \) in equilibrium, by Lemma 1. Proceed again by contradiction and denote with \( \hat{c}_i \) the type at which the jump of \( g_{1i} \) from \( \bar{x} \) to \( \gamma_l \) occurs. Given Step 1, there are two cases to consider. In the first, \( (\gamma_l, \bar{x}) \subset \text{supp} \, H_1^j \) for all other agents \( t \neq i \) in group 1. In the second, there exists some other agent \( k \) in group 1 with \( H_1^k \) putting zero probability on \( (\gamma_h, \bar{x}) \), but strictly positive probability on \( (\gamma_h - \varepsilon, \gamma_h) \).
The first case is handled as in Lemma 1. Indeed, if one starts from condition (29), substituting $\bar{x}$ for $\gamma_h$, then all other steps leading to a contradiction follow unchanged.

In the second case, we have $U_i(\bar{x}; \hat{c}_i) \geq U_i(\gamma_h; \hat{c}_i)$. Furthermore, letting $\hat{c}_k$ denote the cost at which $g_{ik}$ jumps from $\bar{x}$ to $\gamma_h$, we see that agent $k$’s indifference between $\gamma_h$ and $\bar{x}$ means $U_k(\gamma_h; \hat{c}_k) = U_k(\bar{x}; \hat{c}_k)$. These conditions, together with (26) and (28), yield

$$0 \leq U_i(\bar{x}; \hat{c}_i) - U_i(\gamma_h; \hat{c}_i)$$

$$= v \int_{\gamma_h}^{\bar{x}} (1 - H_1^{-\{i\}}(s)) \, dH_2(s) - \hat{c}_i(\bar{x} - \gamma_h)$$

$$= v \int_{\gamma_h}^{\bar{x}} (1 - H_1^{-\{i,k\}}(s))(1 - H_1^{\hat{c}_k}(s)) \, dH_2(s) - \hat{c}_i(\bar{x} - \gamma_h)$$

$$= vF_1(\hat{c}_k) \int_{\gamma_h}^{\bar{x}} (1 - H_1^{-\{i,k\}}(s)) \, dH_2(s) - \hat{c}_i(\bar{x} - \gamma_h)$$

(36) \hspace{1cm} \text{diff1}

and

$$0 = U_k(\bar{x}; \hat{c}_k) - U_k(\gamma_h; \hat{c}_k)$$

$$= vF_1(\hat{c}_i) \int_{\gamma_h}^{\bar{x}} (1 - H_1^{-\{i,k\}}(s)) \, dH_2(s) - \hat{c}_k(\bar{x} - \gamma_h).$$

(37) \hspace{1cm} \text{diff2}

Subtracting (37) from (36), we obtain

$$0 \leq (F_1(\hat{c}_k) - F_1(\hat{c}_i))v \int_{\gamma_h}^{\bar{x}} (1 - H_1^{-\{i,k\}}(s)) \, dH_2(s) + (\hat{c}_k - \hat{c}_i)(\bar{x} - \gamma_h).$$

This inequality implies $\hat{c}_i \leq \hat{c}_k$. Moreover, note that $i$ should not want to contribute in $(\gamma_l, \gamma_l + \epsilon)$, and that Lemma 1 shows that there must exist some agent $j$ other than $i$ such that $(\gamma_l, \gamma_l + \epsilon)$ is contained in the support of $H_1^j$. As in the proof of Step 1, $\lim_{\gamma_l \downarrow \gamma_l} \frac{\partial}{\partial \gamma} U_i(\gamma; \hat{c}_i) \leq 0$, (27), and (30) imply $\hat{c}_i \geq \hat{c}_j \equiv \lim_{\gamma_l \downarrow \gamma_l} g_{ij}^{-1}(\gamma)$. We then reach condition (32), which leads to the same contradiction as in Step 1.

We now show that the lowest contribution of all agents is zero.

**Step 3.** For any agent $i$ in group 1, we have $\lim_{\gamma_l \downarrow \gamma_l} g_{ij}(c) = 0$.

**Proof.** Proceed by contradiction and suppose $g_{ij}(\bar{c}) = \gamma_l > 0$. Since $H_1$ puts strictly positive probability on $(\gamma_l - \epsilon, \gamma_l)$, there must exist some agent $j$ contributing in (a dense subset of) this interval. Let $\hat{c}_j \equiv \lim_{\gamma_l \downarrow \gamma_l} g_{ij}^{-1}(\gamma)$ and note that $\hat{c}_j < \bar{c}$. As in the proof of Step 1,

$$0 \leq \lim_{\gamma_l \downarrow \gamma_l} \frac{\partial}{\partial \gamma} U_i(\gamma; \bar{c}) = \lim_{\gamma_l \downarrow \gamma_l} \nu(1 - H_1^{-\{i,j\}}(\gamma))(1 - H_1^j(\gamma)) \, dH_2(\gamma) - \bar{c}$$

31
= v(1 - H_i^{-1}(\gamma))F_1(\hat{c}_j) dH_2(\gamma) - \bar{c}.

At the same time, agent $j$’s FOC yields

$$0 = \lim_{\gamma \uparrow \gamma_l} \frac{\partial}{\partial \gamma} U_j(\gamma; \hat{c}_j) = \lim_{\gamma \uparrow \gamma_l} v(1 - H_i^{-1}(\gamma))(1 - H_i(\gamma)) dH_2(\gamma) - \hat{c}_j$$

$$= v(1 - H_i^{-1}(\gamma)) dH_2(\gamma) - \hat{c}_j$$

$$= \frac{\bar{c}}{F_1(\hat{c}_j)} - \hat{c}_j$$

(by (38))

$$> 0,$$

which is impossible. Therefore, it must be that $g_{1i}(\bar{c}) = 0$. \qed

**Step 4.** If $g_{1i}$ and $g_{1j}$ take value on a common interval $(\gamma_l, \gamma_h)$, then $g_{1i} = g_{1j}$ on this interval, except for a set of measure zero.

**Proof.** By contradiction, consider any $\gamma \in (\gamma_l, \gamma_h)$ such that $\gamma = g_{1i}(\hat{c}_i) = g_{1j}(\hat{c}_j)$, but $\hat{c}_j > \hat{c}_i$. Almost always, the first-order conditions $\frac{\partial}{\partial \gamma} U_j(\gamma; \hat{c}_i) = \frac{\partial}{\partial \gamma} U_j(\gamma; \hat{c}_j) = 0$ must hold; equations (27) and (28) then imply, dividing one FOC by the other, that

$$\frac{\hat{c}_i}{\hat{c}_j} = \frac{1 - H_i(\gamma)}{1 - H_i(\gamma)} = \frac{F_1(\hat{c}_j)}{F_1(\hat{c}_i)},$$

or $\hat{c}_j F(\hat{c}_j) = \hat{c}_i F(\hat{c}_i)$, in contradiction of the assumption that $\hat{c}_j > \hat{c}_i$. \qed

Steps 1, 2, and 3 imply that for any two agents $i$ and $j$ in group 1, the support of $H_i^j$ and $H_i^l$ are the same; by Lemma 1 they equal $S$, and $S$ is either $[0, \bar{x}]$, or $[0, a] \cup \{\bar{x}\}$, for some $a \in (0, \bar{x})$. Step 4 then establishes that identical teammates use essentially the same strategy, which concludes the proof of Lemma 2. \qed

**Proof of Lemma 3.** By Lemma 2 we know players within a team use the same strategy. Let $g_l$ denote the common strategy of players on team $l$, $l = 1, 2$. Moreover, for a nondegenerate equilibrium the range of the strategies is common and the range where they are strictly decreasing is common. Let $\gamma$ be such that in a neighborhood of $g_l^{-1}(\gamma)$ strategy $g_l$ is strictly decreasing, $l = 1, 2$.

Consider player 11 in group 1. Define $c_1^M = \max\{c_{12}, \ldots, c_{1m}\}$, which has the associated cdf $Q(c) = [F(c)]^{n-1}$ and density $q(c) \equiv Q'(c)$. For player 11 in group 1 with cost $c_{11}$ acting like a type $c_{11}^M$, the associated payoff is

$$V_{11}(c_{11}^M, c_{11}) = -c_{11} g_1(c_{11}^M) + \int_0^{c_{11}^M} v \cdot \text{Pr}(g_2(c_{2j}) < g_1(c_{11}^M), \text{ for some } j = 1, \ldots, n) q(c_{11}^M) dc_{11}^M$$

32
A similar analysis for player 21 in group 2 yields the condition

\[ + \int_{c_{11}}^c v \cdot \Pr(g_2(c_2) < g_1(c_1^M), \text{ for some } j = 1, \ldots, n) \ q_1(c_1^M) \ dc_1^M \]

\[ = -c_{11} g_1(c_{11}^a) + v Q(c_{11}^a) [1 - \Pr(g_2(c_2) \geq g_1(c_{11}^a), \text{ for all } j = 1, \ldots, n)] \]

\[ + \int_{c_{11}}^c v \cdot \Pr(g_2(c_2) \geq g_1(c_1^M), \text{ for all } j = 1, \ldots, n) \ q_1(c_1^M) \ dc_1^M \]

\[ = -c_{11} g_1(c_{11}^a) + v Q(c_{11}^a) \left\{ 1 - \left[ F(g_2^{-1}(g_1(c_{11}^a))) \right]^n \right\} \]

\[ + v \int_{c_{11}}^c \left\{ 1 - \left[ F(g_2^{-1}(g_1(c_1^a))) \right]^n \right\} q(c_1^M) \ dc_1^M. \]

The relevant partial derivative is

\[
\frac{\partial V_{11}}{\partial c_{11}} = -g'_1(c_{11}^a) \left[ c_{11} + n v_1 \left[ F(c_{11}) \right]^{n-1} \left[ F(g_2^{-1}(g_1(c_{11}^a))) \right]^{n-1} \ f(g_2^{-1}(g_1(c_{11}^a))) \ (g_2^{-1})' (g_1(c_{11}^a)) \right].
\]

Where \( g_1 \) is strictly decreasing, in equilibrium \( c_{11}^e = c_{11} \), so the FOC \( \partial V_{11}/\partial c_{11} |_{c_{11}^e=c_{11}} = 0 \) implies

\[
v \left[ F(c_{11}) \right]^{n-1} \left[ F(g_2^{-1}(g_1(c_{11}^a))) \right]^{n-1} \ f(g_2^{-1}(g_1(c_{11}^a))) \ (g_2^{-1})' (g_1(c_{11}^a)) + c_{11} = 0.
\]

A similar analysis for player 21 in group 2 yields the condition

\[
v \left[ F(g_1^{-1}(g_2(c_{21}))) \right]^{n-1} \left[ F(c_{21}) \right]^{n-1} \ f(g_1^{-1}(g_2(c_{21}))) \ (g_1^{-1})' (g_2(c_{21})) + c_{21} = 0.
\]

Following Amann and Leininger (1996), where \( g_1 \) is strictly decreasing we define the function \( \xi(c_{11}) = g_2^{-1}(g_1(c_{11})) \); thus, if a group-1 player with cost \( c' \) exerts effort \( \gamma \), then so too, does a group-2 player with cost \( \xi(c') \). Note that where \( g_1 \) and \( g_2 \) are strictly decreasing, their inverse functions \( g_1^{-1} \) and \( g_2^{-1} \) are well-defined and decreasing. And since both \( g_1 \) and \( g_2 \) are decreasing, \( \xi \) is increasing. By the chain rule,

\[
\xi’(c) = (g_2^{-1})'(g_1(c))g_1'(c),
\]

so \((g_2^{-1})'(g_1(c_{11})) = \xi'(c_{11})/g_1'(c_{11}). \) Now the FOC (41) can be rewritten as

\[
nv F(c_{11})^{n-1} F(\xi(c_{11}))^{n-1} f(\xi(c_{11})) \xi'(c_{11}) = -c_{11} g_1'(c_{11})
\]

For \( c_{21} = \xi(c_{11}) = g_2^{-1}(g_1(c_{11})) \), the second FOC (42) can be rewritten as

\[
fv F(c_{11})^{n-1} F(\xi(c_{11}))^{n-1} f(c_{11}) = -\xi(c_{11}) g_1'(c_{11}).
\]
From (44), solve for $g_1'(c_{11})$ and substitute the solution into (54) to obtain

$$c_{11} f(c_{11}) = \xi(c_{11}) f(\xi(c_{11})) \zeta(c_{11}). \quad (45)$$

Let $\bar{x} \equiv g_1(c) = g_2(c)$. For any $c \in [\underline{c}, \bar{c}]$, define

$$M(c) = \int_{\underline{c}}^{c} x f(x) \, dx.$$ 

Observe that $M(\cdot)$ is nonnegative, finite, and strictly increasing on the support $[\underline{c}, \bar{c}]$.

Now there are two cases to consider, either $g_1$ and $g_2$ place no mass at $\bar{x}$ or they do place mass at $\bar{x}$. First, suppose the equilibrium places no mass at $\bar{x}$, so $\xi(c) = c$. Suppose $g_1$ is strictly decreasing at $c_{11}$. Then, integrating both sides of (45), we have

$$M(c_{11}) = \int_{\underline{c}}^{c_{11}} x f(x) \, dx = \int_{\underline{c}}^{c_{11}} \xi(x) f(\xi(x)) \zeta'(x) \, dx$$

$$= \int_{\underline{c}}^{\xi(c_{11})} y f(y) \, dy \quad \text{(where $y = \xi(x)$)}$$

$$= M(\xi(c_{11})),$$

because $M$ is strictly increasing, it follows that $\xi(c_{11}) = c_{11}$, in turn implying $g_1(c) = g_2(c)$ for all $c$. If instead each strategy places mass at $\bar{x}$, then there is some $\tilde{c}_l > \underline{c}$ such that $g_l(c) = \bar{x}$ for all $c < \tilde{c}_l$ and $\lim_{c \to \tilde{c}_l} g_l(c) = x'$ for some $x' < \bar{x}$, $l = 1, 2$ (see Lemma 1). A group-$l$ player with type $\tilde{c}_l$ must be indifferent between efforts $\bar{x}$ and $x'$. From (39), a group-$1$ player with type $\tilde{c}_1$ has payoff

$$U_1(x'; \tilde{c}_1) = -\tilde{c}_1 x' + vQ(\tilde{c}_1) \left\{ 1 - [F(\tilde{c}_2)]^n \right\} + v \int_{\tilde{c}_1}^{\bar{x}} \left\{ 1 - [F(\xi(c_{11}^M))]^n \right\} q(c_{11}^M) \, dc_{11}^M.$$ 

if he exerts effort $x'$, and he has payoff

$$U_1(\bar{x}; \tilde{c}_1) = -\tilde{c}_1 \bar{x} + vQ(\tilde{c}_1) \left\{ 1 - \frac{1}{2} [F(\tilde{c}_2)]^n \right\} + v \int_{\tilde{c}_1}^{\bar{x}} \left\{ 1 - [F(\xi(c_{11}^M))]^n \right\} q(c_{11}^M) \, dc_{11}^M$$

if he exerts effort $\bar{x}$ (the “1/2” arises because, if all players exert effort $\bar{x}$, each team wins with probability 1/2). Indifference between $x'$ and $\bar{x}$ now implies

$$\tilde{c}_1(\bar{x} - x') = \frac{v}{2} Q(\tilde{c}_1) F(\tilde{c}_2) n = \frac{v}{2} F(\tilde{c}_1)^{n-1} F(\tilde{c}_2)^n.$$
For members of group 2, analogous reasoning yields

\[ \tilde{c}_2(\bar{x} - x') = \frac{v}{2} F(\tilde{c}_1)^n F(\tilde{c}_2)^{n-1}. \]

Therefore,

\[ \tilde{c}_1(\bar{x} - x') F(\tilde{c}_1) = \frac{v}{2} F(\tilde{c}_1)^n F(\tilde{c}_2)^n = \tilde{c}_2(\bar{x} - x') F(\tilde{c}_2), \]

which, because \( F \) is strictly increasing, in turn implies \( \tilde{c}_2 = \tilde{c}_1 \). Let \( \tilde{c} \equiv \tilde{c}_1 \). Using the condition \( \xi(\tilde{c}) = \tilde{c} \), we can proceed as above to integrate (45) from \( \tilde{c} \) to \( c \) to conclude that \( \xi(c) = c \) for all \( c > \tilde{c} \). It then follows that \( g_1(c) = g_2(c) \) for all \( c > \tilde{c} \); and for all \( c < \tilde{c} \) it is the case that \( g_1(c) = g_2(c) = \bar{x} \). \( \square \)

**Proof of Proposition 1.** Expected cost is

\[
\int_{\xi}^{\tilde{c}} cg(c) f(c) \, dc = \int_{\xi}^{\tilde{c}} c \left[ \frac{vn}{2} \int_{c}^{\tilde{c}} F(\tau)^{2(n-1)} f(\tau) \, d\tau \right] f(c) \, dc \quad \text{(by (2))}
\]

\[
= v \int_{\xi}^{\tilde{c}} \frac{n}{2} \int_{\xi}^{\tilde{c}} c f(c) \, dc \frac{F(\tau)^{2n-1} f(\tau)}{\tau} \, d\tau \quad \text{(reversing the order of integration)}
\]

\[
= v \int_{\xi}^{\tilde{c}} \frac{n}{2} W(\tau) d[F(\tau)^{2n}]
\]

\[
= v \int_{\xi}^{\tilde{c}} \frac{1}{2} W(\tau) d[F(\tau)^{2n}].
\]

Since \( F(\tau)^{2(n+1)} \) first-order stochastically dominates \( F(\tau)^{2n} \), if \( W(\tau) \) is increasing in \( \tau \), then expected cost is larger with \( n + 1 \) agents than with \( n \). Therefore each agent’s expected utility is smaller with more agents per group. The rest of the proof follows along the same lines and is here omitted. \( \square \)

**Verification of equation (9).** Starting with

\[ EC(c_0) = \int_{\xi}^{c_0} c\bar{x}(c_0) f(c) \, dc + \int_{c_0}^{\tilde{c}} cg(c) f(c) \, dc, \]

we obtain

\[
EC'(c_0) = c_0 \bar{x}(c_0) f(c_0) + \bar{x}'(c_0) \int_{\xi}^{c_0} c f(c) \, dc - c_0 g(c_0) f(c_0)
\]

\[
= c_0 f(c_0)(\bar{x}(c_0) - g(c_0)) + \bar{x}'(c_0) \int_{\xi}^{c_0} c f(c) \, dc
\]

\[
= f(c_0) \frac{v}{2} [F(c_0)]^{2n-1} + \left[ g'(c_0) + \frac{v}{2c_0} (2n - 1) [F(c_0)]^{2n-2} f(c_0) - \frac{v}{2(c_0)^2} [F(c_0)]^{2n-1} \right] \int_{\xi}^{c_0} c f(c) \, dc
\]

35
\[
\begin{align*}
&= f(c_0) \frac{v}{2} [F(c_0)]^{2n-1} - \left[ \frac{v}{2c_0} [F(c_0)]^{2n-2} f(c_0) + \frac{v}{2(c_0)^2} [F(c_0)]^{2n-1} \right] \int_{c_0}^{\infty} c f(c) \, dc \\
&= \frac{v}{2} [F(c_0)]^{2n-2} \left[ f(c_0) F(c_0) - \left( \frac{f(c_0)}{c_0} + \frac{F(c_0)}{(c_0)^2} \right) \int_{c_0}^{\infty} c f(c) \, dc \right].
\end{align*}
\]

Proof of Proposition 2. Using (4), we have

\[
W'(\tau) = \frac{\tau f(\tau) F(\tau) - [F(\tau) + \tau f(\tau)] \int_{0}^{\tau} c f(c) \, dc}{(\tau F(\tau))^2};
\]

therefore, if \( W'(c_0) > 0 \), then the term in square brackets in (9) is positive, so expected cost increases in \( c_0 \).

Since in a symmetric equilibrium expected utility and cost move in opposite directions, we have established that if \( W'(\tau) > 0 \), then expected utility decreases in \( c_0 \). The rest of the proof follows along similar lines. \( \square \)

Proof of Theorem 5. Consider player 11 in group 1. Let \( c_1^M = \max\{c_{12}, \ldots, c_{1n_1}\} \), with cdf \( H_1 \). So \( H_1(c) = [F_1(t)]^{n_1-1} \). For player 11 in group 1 with cost \( c_{11} \) announcing type \( c_{11}^a \) the associated payoff is

\[
V_{11}(c_{11}^a, c_{11}) = -c_{11} g_1(c_{11}^a) + \int_{c_{11}}^{c_1^M} v_1 \cdot \Pr(g_2(c_{2j}) < g_1(c_{11}^a), \text{ for some } j = 1, \ldots, n_2) \ h_1(c_1^M) \, dc_1^M \\
+ \int_{c_{11}}^{1} v_1 \cdot \Pr(g_2(c_{2j}) < g_1(c_1^a), \text{ for some } j = 1, \ldots, n_2) \ h_1(c_1^M) \, dc_1^M
\]

\[
= -c_{11} g_1(c_{11}^a) + v_1 H_1(c_{11}^a) \left[ 1 - \Pr(g_2(c_{2j}) \geq g_1(c_{11}^a), \text{ for all } j = 1, \ldots, n_2) \right]
\]

\[
+ \int_{c_{11}}^{1} v_1 \left[ 1 - \Pr(g_2(c_{2j}) \geq g_1(c_1^a), \text{ for all } j = 1, \ldots, n_2) \right] h_1(c_1^M) \, dc_1^M
\]

\[
= -c_{11} g_1(c_{11}^a) + v_1 H_1(c_{11}^a) \left\{ 1 - \left[ F_2(g_2^{-1}(g_1(c_{11}^a))) \right]^{n_2} \right\}
\]

\[
+ v_1 \int_{c_{11}}^{1} \left\{ 1 - \left[ F_2(g_2^{-1}(g_1(c_{11}^a))) \right]^{n_2} \right\} h_1(c_1^M) \, dc_1^M.
\]

The relevant partial derivative is

\[
\frac{\partial V_{11}}{\partial c_{11}^a} = -g_1'(c_{11}^a) \left[ c_{11} + n_2 v_1 \left[ F_1(c_{11}) \right]^{n_1-1} \left[ F_2(g_2^{-1}(g_1(c_{11}^a))) \right]^{n_2-1} f_2(g_2^{-1}(g_1(c_{11}^a))) \left( g_2^{-1} \right)'(g_1(c_{11}^a)) \right].
\]

In equilibrium \( c_{11}^a = c_{11} \), so we rewrite the FOC \( \partial V_{11}/\partial c_{11}^a|_{c_{11}^a=c_{11}} \) as

\[
n_2 v_1 \left[ F_1(c_{11}) \right]^{n_1-1} \left[ F_2(g_2^{-1}(g_1(c_{11}^a))) \right]^{n_2-1} f_2(g_2^{-1}(g_1(c_{11}^a))) \left( g_2^{-1} \right)'(g_1(c_{11}^a)) + c_{11} = 0.
\]

36
Similarly, write $V_{21}(c_{21}, c_{21})$ for player 1 in group 2 and take the FOC to get

$$n_1 v_2 \left[ F_1(g_1^{-1}(g_2(c_{21}))) \right]^{n_1-1} [F_2(c_{21})]^{n_2-1} f_1(g_1^{-1}(g_2(c_{21}))) (g_1^{-1})' (g_2(c_{21})) + c_{21} = 0. \quad (48)$$

Observe that $(g_2^{-1})'(g_1(c_{11})) = \xi'(c_{11})/g_1'(c_{11})$ and obtain from (47) the first optimality condition:

$$n_2 v_1 F_1(c_{11})^{n_1-1} F_2(\xi(c_{11}))^{n_2-1} f_2(\xi(c_{11})) \xi'(c_{11}) = -c_{11} g_1'(c_{11}). \quad (49)$$

For $c_{21} = g_2^{-1}(g_1(c_{11})) = \xi(c_{11})$, (48) yields the second optimality condition:

$$n_1 v_2 F_2(\xi(c_{11}))^{n_2-1} F_1(c_{11})^{n_1-1} f_1(c_{11}) = -\xi(c_{11}) g_1'(c_{11}). \quad (50)$$

From (50), express $g_1'(c_{11})$ and substitute into (49), yielding

$$\xi'(c_{11}) = \frac{n_1 v_2 c_{11} f_1(c_{11})}{n_2 v_1 \xi(c_{11}) f_2(\xi(c_{11}))}. \quad (51)$$

Let us introduce a change of variable $\chi(c_{11}) \equiv \xi^2(c_{11})$, then $\chi'(c_{11}) = 2\xi(c_{11})\xi'(c_{11})$, and equation (51) can be written as

$$\chi'(c_{11}) = \frac{2n_1 v_2 c_{11} f_1(c_{11})}{n_2 v_1 f_2(\sqrt{\chi(c_{11})})}. \quad (52)$$

Note that since no group puts mass at $\bar{x}$, by Part 3 of Lemma 1 we have $\xi(\bar{x}) = \bar{c}$ and, therefore, $\chi(\bar{x}) = \bar{c}$. Equation (52) together with the initial condition $\chi(\bar{x}) = \bar{c}$ constitute an initial value problem. The Lipschitz condition is satisfied at $\bar{c}$, provided $\lim_{x \to \bar{c}} f_2(x) \neq 0$. Therefore, under this condition there exists a unique solution for $\chi$ by the Picard-Lindelöf theorem (Coddington and Levinson, 1955, Theorem 3.1, p.12). Since $\xi$ is nonnegative, there exists a unique solution for $\xi$, defined as $\xi(c_{11}) = \sqrt{\chi(c_{11})}$. Next, we solve for $g_1$ from (50) as

$$g_1(c_{11}) = g_1(c_{11}) - g_1(\bar{c}) = -\int_{c_{11}}^{\bar{c}} g_1'(\tau)d\tau \quad (53) \quad \text{(xsol)}$$

Finally, solve for $g_2$ from

$$g_2(c_{21}) = g_1 \left[ \xi^{-1}(c_{21}) \right]. \quad \square$$

Proof of Proposition 5. We begin by deriving equilibrium, along the lines of Theorem 5, and we show that the probability of winning for group 1 increases with respect to the symmetric case in Theorem 3. Consider

37
the utility of an agent in group 1 (the organized group) with cost type \( c \) that pretends to be \( c' \).

\[
V_o(c', c) = -c \left( \int_{\xi}^{c'} g_1(c')dF^{n-1}(z) + \int_{c'}^{\bar{\xi}} g_1(z)dF^{n-1}(z) \right) \\
+ v \left( \int_{\xi}^{c'} (1 - F^n (g_2^{-1}(g_1(c')))) dF^{n-1}(z) + \int_{c'}^{\bar{\xi}} (1 - F^n (g_2^{-1}(g_1(z)))) dF^{n-1}(z) \right).
\]

The only difference with the formulation in Theorem 5 is that agents only end up contributing the amount preferred by the highest-cost type. The derivative w.r.t. \( c' \) gives

\[
\frac{\partial V_o(c', c)}{\partial c'} = g'_1(c')F^{n-1}(c') \left[ -c - vnF^{n-1} (g_2^{-1}(g_1(c'))) f (g_2^{-1}(g_1(c'))) \frac{1}{g_2'(g_2^{-1}(g_1(c')))} \right].
\]

When the strategy is strictly decreasing, the truth-telling FOC (which is necessary and sufficient for the same reason described in Theorem 5) is

\[
c = -vnF^{n-1} (g_2^{-1}(g_1(c'))) f (g_2^{-1}(g_1(c'))) \frac{1}{g_2'(g_2^{-1}(g_1(c')))}.
\]

Since \( \xi(c) \equiv g_2^{-1}(g_1(c)) \), we have \( \xi'(c) = \frac{g'_1(c)}{g_2'(g_2^{-1}(g_1(c)))} \), so the above displayed equation can be rewritten as

\[
c \cdot g'_1(c) = -\xi'(c) vnF^{n-1} (\xi(c)) f (\xi(c)); \tag{54}
\]

this is the analogue to equation (49). Consider now the utility of an agent in group 2 with cost type \( c \) that pretends to be \( c' \):

\[
V_2(c', c) = -cg_2(c') + v \left( \int_{\xi}^{c'} (1 - F^n (g_1^{-1}(g_2(c')))) dF^{n-1}(z) + \int_{c'}^{\bar{\xi}} (1 - F^n (g_1^{-1}(g_2(z)))) dF^{n-1}(z) \right).
\]

This is the same expression we derived in the Proof of Theorem 5. Proceeding in the same fashion as done there, the truth-telling FOC, evaluated at \( \xi(c) \), yields

\[
\xi(c) g'_1(c) = -vnF^{n-1} (\xi(c)) F^{n-1}(c) f (c),
\]

which is exactly (50). Dividing (54) by (50) and rearranging, we obtain

\[
\xi'(c) = \frac{cf (c)}{\xi(c) f (\xi(c))} \cdot F^{n-1}(c), \tag{55}
\]

which should be compared to (51), which if the teams are symmetric yields \( \xi(c) = c \). Following the same
logic as in the Proof of Proposition 3, we see that after group 1 gets organized, (55) leads to \( \xi(c) < c \) for any \( c \in (\xi, \tilde{c}) \). Intuitively, that is because at any putative point \( c_0 \in (\xi, \tilde{c}) \) with \( \xi(c_0) = c_0 \), we obtain \( \xi'(c_0) < 1 \), which immediately pushes \( \xi(c) \) below the 45 degree line. However, we cannot follow the same logic of the Proof of Proposition 3 to conclude also that \( \xi(\tilde{c}) < \tilde{c} \) because this would require \( \xi'(\tilde{c}) \geq 1 \), but this is now not contradicted by (55) under \( \xi(\tilde{c}) = \tilde{c} \), which yields \( \xi'(\tilde{c}) = 1 \). Nonetheless, we can use another method of proof to show \( \xi(\tilde{c}) < \tilde{c} \), if \( f(\tilde{c}) > 0 \), which we have assumed in this section. Suppose to the contrary that \( \xi(\tilde{c}) = \tilde{c} \). (\( \xi(\tilde{c}) > \tilde{c} \) is impossible because \( \xi(c) < c \) for any \( c \in (\xi, \tilde{c}) \) and \( \xi \) is continuous.) Using (55) we then have \( \xi'(\tilde{c}) = 1 \), and differentiating (55) we obtain \( \xi''(\tilde{c}) = (n - 1)f(\tilde{c}) > 0 \). Define \( \Delta(c) \equiv \xi(c) - c \). Note that \( \Delta(\tilde{c}) = \Delta'(\tilde{c}) = 0 \) under the hypothesis \( \xi(\tilde{c}) = \tilde{c} \). A second-order Taylor expansion now yields

\[
\Delta(c) - \Delta(\tilde{c}) \approx \Delta'(\tilde{c})(c - \tilde{c}) + \frac{1}{2} \Delta''(\tilde{c})(c - \tilde{c})^2 = \frac{1}{2}(n - 1)f(\tilde{c})(c - \tilde{c})^2 > 0.
\]

Therefore, the above-displayed equation gives \( \xi(c) - c > 0 \) for some \( c \) in a left neighborhood of \( \tilde{c} \), a contradiction to \( \xi(c) < c \). We conclude that for any type \( c > \xi \), the probability of victory of group 1 is strictly larger after group 1 gets organized.

We now compare the interim equilibrium utility of an agent in the organized group, \( W_o(c) \equiv V_o(c, c) \), with that of an agent in the symmetric equilibrium of Theorem 3, \( W_{no}(c) \equiv V_{11}(c, c) \), as defined in (3), and we establish that \( W_o(c) > W_{no}(c) \), for all \( c > \xi \). We begin with \( \lim_{c \to \xi} W_o(c) = v(1 - F^n(\xi)) > 0 = W_{no}(\tilde{c}) \).

Now suppose that by contradiction there exists some \( \tilde{c} \) such that \( W_o(\tilde{c}) = W_{no}(\tilde{c}) \), with \( W_o(c) > W_{no}(c) \) for all \( c > \tilde{c} \). Then, using the envelope theorem, we must have

\[
\int_{\xi}^{\tilde{c}} g_1(\tilde{c})dF_{n-1}(z) + \int_{\tilde{c}}^{\tilde{c}} g_1(z)dF_{n-1}(z) = |W'_o(\tilde{c})| \leq |W'_{no}(\tilde{c})| = g(\tilde{c}),
\]

where \( g \) is the equilibrium contribution function in the symmetric equilibrium of Theorem 3. But this contradicts \( W_o(\tilde{c}) = W_{no}(\tilde{c}) \), since it implies that \( \tilde{c} \)'s contribution under organization (and hence its cost) is weakly lower than if neither group is organized, and we have previously established that the probability of winning is strictly larger under organization, so \( W_o(\tilde{c}) > W_{no}(\tilde{c}) \).

\( \square \)

Proof of Proposition 3. Let \( \frac{n}{n_1} < \frac{n}{n_2} \cdot \frac{f_1(c)}{f_2(c)} \). We begin by showing that \( \xi(c) > c \ \forall c \in (\xi, \tilde{c}) \). Note that \( \xi'(c) = \frac{n_1 v_2 f_1(c)}{n_2 v_1 f_2(c)} > 1 \), therefore by continuity of \( \xi \), there exists \( \epsilon > 0 \) such that \( \xi(c) > c \) for all \( c \in (\xi, \xi + \epsilon) \).

Suppose by contradiction \( \xi(c_1) \leq c_1 \) for some \( c_1 > \xi \). By continuity of \( \xi \), then there exists some smallest cost \( c^* \) such that \( \xi(c^*) = c^* \). Since \( \xi(c) > c \ \forall c \in (\xi, c^*) \), a necessary condition for \( \xi(c^*) = c^* \) is \( \xi'(c^*) \leq 1 \). But (11) implies \( \xi'(c^*) = \frac{n_1 v_2 f_1(c^*)}{n_2 v_1 f_2(c^*)} > 1 \), a contradiction. Therefore \( \xi(c) > c \) for all \( c \in (\xi, \tilde{c}) \). We derive two implications. First, since \( \xi(c) = g_2^{-1}(g_1(c)) \) and \( g_2 \) is strictly decreasing, applying \( g_2 \) to both sides of
we obtain \( g_1(c) < g_2(c) \). Second, \( \xi(\bar{c}) > \bar{c} \). Therefore, by continuity there exists \( c' < \bar{c} \) such that \( \xi(c') = \bar{c} \), so group-1-player types in \([c', \bar{c}]\) never win. Therefore, these types exert zero effort. The results for the remaining two cases can be proven analogously.

\[\square\]

**Proof of Proposition 4.** Fix an equilibrium with mass at the highest effort, with the corresponding cutoffs \( c_01 \) and \( c_02 \). Assume \( \frac{f_1(c_01)}{f_2(c_01)} > \frac{n_1}{n_2} \) and \( \frac{n_2}{n_1 n_2} < \frac{c_01}{f_2(c_01)} \) for all \( c \geq c_01 \). We now show \( \xi(c) > c \) for all \( c \in (c_01, \bar{c}] \). Using the initial condition of Theorem 6, observe that \( c_02 > c_01 \) if and only \( \nu_2^{-1}\left(\frac{c_01}{\nu_1 c_01 f_1(c_01)}\right) > c_01 \). Since \( \nu_2^{-1} \) is increasing, this inequality is equivalent to

\[
\frac{\nu_2}{\nu_1} c_01 F_1(c_01) > \nu_2(c_01) = c_01 F_2(c_01),
\]

which is equivalent to

\[
\frac{F_1(c_01)}{F_2(c_01)} > \frac{\nu_1}{\nu_2}.
\]

Therefore, our first assumption implies \( c_02 > c_01 \), thus \( \xi(c_01) > c_01 \). Note that \( \xi'(c_01) = \frac{n_1 \nu_2 f_1(c_01)}{n_2 \nu_1 f_2(c_01)} > 1 \), therefore by continuity of \( \xi \), there exists \( \epsilon > 0 \) such that \( \xi(c) > c \) for all \( c \in (c_01, c_01 + \epsilon] \). The rest of the proof now follows just as that of Proposition 3. \[\square\]

**References**


Parreiras, Sergio, and Anna Rubinchik (2010), Contests with three or more heterogeneous agents, *Games and Economic Behavior* 68 (2), 703–715.

